General frames for interpretability logic **IL**

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Introduction

General frames (for normal modal logics):

- Kripke frames with additional structure,
- combine "nice" properties of Kripke and algebraic semantics; intuition and completeness.

Interpretability logics:

- extension of provability logic **GL**,
- interpreted on Kripke-like frames called Veltman frames.

Goals of this talk:

- define general frames for interpretability logics,
- check the similarity between the properties of general frames for modal and interpretability logics.

General frames

- general frame: $(\mathfrak{F}, A)$, where $\mathfrak{F} = (W, R)$ is a Kripke frame, and A a set of subsets of W satisfying some closure properties,
- every Kripke frame is a general frame $(A = \mathcal{P}(W))$,
- if $(\mathfrak{F}, V)$ is a Kripke model, then for $A = \{V(\varphi) : \varphi$ a modal formula$\}$, $(\mathfrak{F}, A)$ is a general frame,
- modal logic **K** is sound and strongly complete with respect to the class of all general frames.

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Logic **IL**

Alphabet of logic **IL** is the union of the following sets:

- a countable set $\mathrm{Prop} = \{p_0, p_1, p_2, \ldots\}$, of propositional variables,
- a set $\{\bot\}$,
- a set $\{\to\}$,
- a set $\{\rhd\}$ and
- a set $\{(,)\}$.

A formula of **IL** is given by the following:

$$\varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \varphi \rhd \varphi,$$

where $p \in \mathrm{Prop}$.

Other symbols

We define $\neg$, $\wedge$, $\vee$, $\leftrightarrow$, $\top$, $\Box$ i $\Diamond$ as follows:

- $\neg\varphi := \varphi \to \bot$,
- $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$,
- $\varphi \vee \psi := \neg\varphi \to \psi$,
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$,
- $\top := \neg\bot$,
- $\Box\varphi := (\neg\varphi) \rhd \bot$ and
- $\Diamond\varphi := \neg\Box\neg\varphi$.

System IL

System **IL** contains all propositional tautologies and all instantiations of the following:

L1 $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,

L2 $\Box\varphi \to \Box\Box\varphi$,

L3 $\Box(\Box\varphi \to \varphi) \to \Box\varphi$,

J1 $\Box(\varphi \to \psi) \to (\varphi \rhd \psi)$,

J2 $((\varphi \rhd \psi) \land (\psi \rhd \chi)) \to (\varphi \rhd \chi)$,

J3 $((\varphi \rhd \chi) \land (\psi \rhd \chi)) \to ((\varphi \lor \psi) \rhd \chi)$,

J4 $(\varphi \rhd \psi) \to (\Diamond\varphi \to \Diamond\psi)$,

J5 $(\Diamond\varphi \rhd \varphi)$.

Rules of inference are:

- modus ponens: from $\varphi \to \psi$ and φ derive ψ,
- necessitation: from φ derive $\Box\varphi$.

Proof

A *proof* of a formula φ in **IL** is a finite sequence of formulae such that φ is the final formula in the sequence and every formula in the sequence is

- a tautology,
- and instantiation of an axiom schema of **IL**,
- derived by a rule of inference from some of the previous formulas.

If there exists a proof of φ, we refer to φ as *provable* in **IL** or a *theorem* of **IL** and denote it as $\vdash \varphi$.

Derivation

A *derivation* of a formula φ from a set Γ in **IL** is a finite sequence of formulae such that φ is the final formula in the sequence and every formula in the sequence is

- theorem of **IL**,
- an element of Γ,
- derived by modus ponens from some of the previous formulas.

If such a derivation exists, we refer to φ as *derivable* from Γ in **IL** and denote it as $\Gamma \vdash_{\mathbf{IL}} \varphi$.

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Veltman Semantics

Definition

A *Veltman frame* $\mathfrak{F}$ is a triple $(W, R, \{S_w : w \in W\})$, where W is a non-empty set, R transitive and conversely well-founded binary relation on W and $\{S_w : w \in W\}$ a collection of binary relations on $R[w]$, where, for all $w \in W$, S_w is a reflexive and transitive and the restriction of R onto $R[w]$ is contained in S_w.

Definition

A *Veltman model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a Veltman frame and $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a *valuation* function.

Veltman Semantics

Valuation indeces a *forcing* relation $\Vdash$, in the following way:

$$
\begin{aligned}
w \Vdash p & \iff w \in V(p), \\
w \Vdash \bot & \quad \text{for no } w \in W, \\
w \Vdash \varphi \to \psi & \iff w \nVdash \varphi \text{ or } w \Vdash \psi, \\
w \Vdash \Box\varphi & \iff \forall v(wRv \Rightarrow v \Vdash \varphi), \\
w \Vdash \varphi \rhd \psi & \iff \forall u\,(wRu \;\&\; u \Vdash \varphi \Rightarrow \exists v(uS_w v \;\&\; v \Vdash \psi))\,.
\end{aligned}
$$

We also write $\mathfrak{M}, w \Vdash \varphi$, if we want to specify the model $\mathfrak{M}$. If for all $w \in W$ in a model $\mathfrak{M}$, $w \Vdash \varphi$ holds, we write $\mathfrak{M} \Vdash \varphi$.
Forcing relation extends the valuation function to a set of all formulae:

$$V(\varphi) = \{w \in W : w \Vdash \varphi\}.$$

Completeness

- F. Veltman, D. de Jongh. *Provability Logics for Relative Interpretability*, Mathematical Logic, Springer, Boston, MA, 1990.
- G. Japaridze, D. de Jongh. *The Logic of Provability*, Handbook of Proof Theory, Elsevier, Amsterdam, 1998.

Theorem (Weak completeness)

If $\nvdash_{\mathbf{IL}} \varphi$, then there exists a finite Veltman model $\mathfrak{M}$ such that $\mathfrak{M} \nVdash \varphi$.

Completeness

Strong completeness, however, does not hold. Consider a set

$$S := \{\Diamond p_0\} \cup \{p_n \rightarrow \Diamond p_{n+1} : n \in \mathbb{N}\}.$$

This set is consistent, because all of its finite subsets are consistent (we can find models which satisfy them). But the set S is not satisfied at any Veltman model. So, even though $S \nvdash_{\mathbf{IL}} \bot$, there is no model which satisfies S and invalidates $\bot$.

Problem: S "needs" an infinite R-chain, which is impossible due to converse well-foundedness of R from the definition of Veltman frame.

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Quasi-Veltman models

Definition

A *quasi-Veltman frame* $\mathfrak{F}$ is a triple $(W, R, \{S_w : w \in W\})$, where W is a non-empty set, R transitive and irreflexive binary relation on W and $\{S_w : w \in W\}$ a collection of binary relations on $R[w]$, where, for all $w \in W$, S_w is a reflexive and transitive and the restriction of R onto $R[w]$ is contained in S_w.

Definition

A *quasi-Veltmanov model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a quasi-Veltman frame and $V : \text{Prop} \to \mathcal{P}(W)$ is a *valuation* function.

Quasi-Veltman models

Completely analogous to Veltman model case, we define the forcing relation and we extend the valuation function to the set of all formulae.

Problem: There exist quasi-Veltman models which do not satisfy $\Box(\Box p \rightarrow p) \rightarrow \Box p$.
Consider $(\mathbb{N}, <, \{\leq|_{<[n]} : n \in \mathbb{N}\}, V)$, where $V(p) = 2\mathbb{N}$. Then

- for no $n \in \mathbb{N}$ does $n \Vdash \Box p$ hold,
- therefore, for all $n \in \mathbb{N}$, $n \Vdash \Box p \rightarrow p$ trivially holds,
- then $n \Vdash \Box(\Box p \rightarrow p)$ obviously holds,
- finally: for no $n \in \mathbb{N}$ does $n \Vdash \Box(\Box p \rightarrow p) \rightarrow \Box p$ hold.

General **IL**-frames

Definition

A *general* **IL**-*frame* is a pair $(\mathfrak{F}, A)$, where $\mathfrak{F}$ is a quasi-Veltman frame and $A \subseteq \mathcal{P}(W)$ a non-empty set of *admissible* subsets of W, closed under the following operations:

(i) *union:* if $X, Y \in A$, then $X \cup Y \in A$,

(ii) *complement:* if $X \in A$, then $W \setminus X \in A$,

(iii) $m_\rhd$*:* if $X, Y \in A$, then $m_\rhd(X, Y) \in A$, where

$$m_\rhd(X, Y) = \{w \in W : \forall u \in X\ (wRu \to \exists v \in Y\ (uS_w v))\},$$

and satisfying the property

(iv) for all $X \in A$, $(W \setminus m_\Box((W \setminus m_\Box(X)) \cup X)) \cup m_\Box(X) = W$ holds.

We use $m_\Box(X)$ as a shorthand for $m_\rhd(W \setminus X, \emptyset)$.

General **IL**-frames

Definition

A *model based on a general* **IL***-frame* is a triple $(\mathfrak{F}, A, V)$, where $(\mathfrak{F}, A)$ is a general **IL**-frame, and $V : \mathrm{Prop} \to A$ is an *admissible valuation*, which means that $V(p) \in A$, for all $p \in \mathrm{Prop}$.

Definition of a forcing relation is analogous to the one in the case of Veltman semantics, as is the extension of the valuation to the set of all formulas.

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Strong Completeness

We shall provide a sketch of the proof of the strong completeness, which is very similar to the proof of weak completeness for Veltman frames.

Definition

A set of formulae Γ is *consistent* if $\Gamma \nvdash_{\mathbf{IL}} \bot$.
If Γ is a consistent set and, for any set Γ', if $\Gamma \subsetneq \Gamma'$, then Γ' is inconsistent, then Γ is a *maximally consistent set*.

Strong Completeness

Definition

Let Γ and Δ be two maximally consistent sets. We say that Δ is a *successor* of Γ, and denote it as $\Gamma \prec \Delta$, if

- for every formula $\Box\varphi \in \Gamma$, $\Box\varphi, \varphi \in \Delta$ holds and
- there exists a formula $\Box\psi \notin \Gamma$ such that $\Box\psi \in \Delta$.

If, additionally,

- $\neg\varphi, \Box\neg\varphi \in \Delta$ for all φ such that $\varphi \rhd \psi \in \Gamma$,

then Δ is a ψ-*critical successor* of Γ.

Strong Completeness

Assume that $\Gamma \nvdash_{\mathbf{IL}} \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is a consistent set. There exists a maximally consistent set Γ' which contains $\Gamma \cup \{\varphi\}$.

We define the following:

- W is the smallest set of pairs $w = (w_0, w_1)$, where w_0 is a maximally consistent set and w_1 a finite sequence of formulae such that
  - $(\Gamma', \langle\rangle) \in W$,
  - if $(w_0, w_1) \in W$, then $(w_0', w_1) \in W$ i $(w_0', w_1 * \langle\psi\rangle) \in W$ for each successor w_0' of w_0 and formula ψ,
- $wRv \iff w_0 \prec v_0$ i $w_1 \subseteq v_1$,
- $uS_w v$ if and only if:
  - $u, v \in R[w]$,
  - $w_1 = u_1 \subseteq v_1$, or $u_1 = w_1 * \langle\psi\rangle * \tau$ i $v_1 = w_1 * \langle\psi\rangle * \sigma$, for some formula ψ and finite sequences of formulae τ and σ, where, if u_0 is a ψ-critical successor of w_0, then v_0 is, too.

Strong Completeness

We have defined a quasi-Veltman frame $\mathfrak{F} = (W, R, \{S_w : w \in W\})$. We define a valuation on this frame as follows:

$$w \in V(p) \iff p \in w_0.$$

By induction over the complexity of formulae, we can prove the following:

$$w \Vdash \psi \iff \psi \in w_0,$$

for all $w \in W$ and formula ψ.

For this quasi-Veltman model $\mathfrak{M} = (\mathfrak{F}, V)$, both $\mathfrak{M}, (\Gamma', \langle\rangle) \Vdash \Gamma$ and $\mathfrak{M}, (\Gamma', \langle\rangle) \nVdash \varphi$ hold.

Strong Completeness

Finally, we define $A = \{V(\varphi) : \varphi \text{ formula}\}$, and we may directly check that $(W, R, \{S_w : w \in W\}, A)$ is a general **IL**-frame. Furthermore, V is admissible, therefore, the following theorem holds:

Theorem (Strong completeness)

Logic **IL** is sound and strongly complete with respect to the class of all general **IL**-frames.

Properties of general **IL**-frames

- Every Veltman frame is a general **IL**-frame,
- If $(\mathfrak{F}, A, V)$ is a model based on a general **IL**-frame, then $V(\varphi) \in A$ for all formulas φ,
- If $(\mathfrak{F}, V)$ is a quasi-Veltman model and if for a set $A = \{V(\varphi) : \varphi \text{ is a formula}\}$ the property (iv), holds, then $(\mathfrak{F}, A)$ is a general **IL**-frame.
- Logic **IL** is sound and **strongly complete** with respect to the class of all general **IL**-frames.

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Concluding remarks

This result

- improves upon Veltman semantics by giving us a frame-based semantics where strong completeness holds,
- generalizes the notion of general frame to the logic **IL**.

Concluding remarks

Future work:

- classes of general frames for extensions of **IL**,
- general frames for Verbrugge semantics,
- algebraic semantics for interpretability logics.

General frames for interpretability logic **IL**

Thank you for your attention!

Literatura

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