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General IL-frames

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General frames for interpretability logic IL

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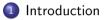
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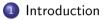
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Introduction

General frames (for normal modal logics):

- Kripke frames with additional structure,
- combine "nice" properties of Kripke and algebraic semantics; intuition and completeness.

Interpretability logics:

- extension of provability logic GL,
- interpreted on Kripke-like frames called Veltman frames.

Goals of this talk:

- define general frames for interpretability logics,
- check the similarity between the properties of general frames for modal and interpretability logics.



General frames

- general frame: (\$\vec{v}\$, A), where \$\vec{v}\$ = (W, R) is a Kripke frame, and A a set of subsets of W satisfying some closure properties,
- every Kripke frame is a general frame $(A = \mathcal{P}(W))$,
- if (ℑ, V) is a Kripke model, then for A = {V(φ) : φ a modal formula}, (ℑ, A) is a general frame,
- modal logic **K** is sound and strongly complete with respect to the class of all general frames.



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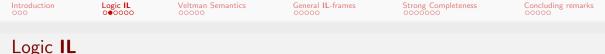
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Alphabet of logic **IL** is the union of the following sets:

- a countable set $\mathsf{Prop} = \{p_0, p_1, p_2, \dots\}$, of propositional variables,
- a set $\{\bot\}$,
- a set $\{\rightarrow\}$,
- \bullet a set $\{\rhd\}$ and
- a set $\{(,)\}$.

A formula of $\ensuremath{\text{IL}}$ is given by the following:

$$\varphi ::= \mathbf{p} \mid \bot \mid \varphi \rightarrow \varphi \mid \varphi \rhd \varphi,$$

where $p \in \mathsf{Prop}$.

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Other symbols

We define \neg , \land , \lor , \leftrightarrow , \top , \Box i \Diamond as follows:

•
$$\neg \varphi := \varphi \to \bot$$
,

•
$$\varphi \wedge \psi := \neg(\varphi \rightarrow \neg \psi)$$
,

$$\bullet \ \varphi \lor \psi := \neg \varphi \to \psi,$$

•
$$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi),$$

•
$$\top := \neg \bot$$
,

•
$$\Box \varphi := (\neg \varphi) \rhd \bot$$
 and

•
$$\Diamond \varphi := \neg \Box \neg \varphi.$$

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System IL					

System IL contains all propositional tautologies and all instantiations of the following:

L1
$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$$
,

L2 $\Box \varphi \rightarrow \Box \Box \varphi$,

L3
$$\Box(\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi$$
,

$$\mathsf{J1} \quad \Box(\varphi \to \psi) \to (\varphi \rhd \psi),$$

$$\mathsf{J2} \quad ((\varphi \rhd \psi) \land (\psi \rhd \chi)) \to (\varphi \rhd \chi),$$

$$\mathsf{J3} \quad ((\varphi \rhd \chi) \land (\psi \rhd \chi)) \to ((\varphi \lor \psi) \rhd \chi),$$

$$\mathsf{J4} \quad (\varphi \rhd \psi) \to (\Diamond \varphi \to \Diamond \psi),$$

J5 (
$$\Diamond \varphi \rhd \varphi$$
).

Rules of inference are:

- \bullet modus ponens: from $\varphi \rightarrow \psi$ and φ derive $\psi \text{,}$
- necessitation: from φ derive $\Box \varphi$.

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Proof					

A *proof* of a formula φ in **IL** is a finite sequence of formulae such that φ is the final formula in the sequence and every formula in the sequence is

- a tautology,
- and instantiation of an axiom schema of IL,
- derived by a rule of inference from some of the previous formulas.

If there exists a proof of φ , we refer to φ as *provable* in **IL** or a *theorem* of **IL** and denote it as $\vdash \varphi$.



A *derivation* of a formula φ from a set Γ in **IL** is a finite sequence of formulae such that φ is the final formula in the sequence and every formula in the sequence is

- theorem of **IL**,
- an element of Γ,

• derived by modus ponens from some of the previous formulas. If such a derivation exists, we refer to φ as *derivable* from Γ in **IL** and denote it as $\Gamma \vdash_{\mathbf{IL}} \varphi$.



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Veltman Semantics

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Definition

A Veltman frame \mathfrak{F} is a triple $(W, R, \{S_w : w \in W\})$, where W is a non-empty set, R transitive and conversely well-founded binary relation on W and $\{S_w : w \in W\}$ a collection of binary relations on R[w], where, for all $w \in W$, S_w is a reflexive and transitive and the restriction of R onto R[w] is contained in S_w .

Definition

A Veltman model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where \mathfrak{F} is a Veltman frame and $V : \operatorname{Prop} \to \mathcal{P}(W)$ is a valuation function.

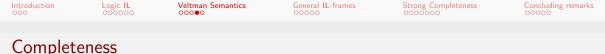


Valuation indeces a *forcing* relation \Vdash , in the following way:

$$\begin{array}{lll} w \Vdash p & \Longleftrightarrow w \in V(p), \\ w \Vdash \bot & \text{for no } w \in W, \\ w \Vdash \varphi \rightarrow \psi & \Longleftrightarrow w \nvDash \varphi \text{ or } w \Vdash \psi, \\ w \Vdash \Box \varphi & \Longleftrightarrow \forall v (wRv \Rightarrow v \Vdash \varphi), \\ w \Vdash \varphi \triangleright \psi & \Longleftrightarrow \forall u (wRu \& u \Vdash \varphi \Rightarrow \exists v (uS_w v \& v \Vdash \psi)). \end{array}$$

We also write $\mathfrak{M}, w \Vdash \varphi$, if we want to specify the model \mathfrak{M} . If for all $w \in W$ in a model $\mathfrak{M}, w \Vdash \varphi$ holds, we write $\mathfrak{M} \Vdash \varphi$. Forcing relation extends the valuation function to a set of all formulae:

$$V(\varphi) = \{ w \in W : w \Vdash \varphi \}.$$



- F. Veltman, D. de Jongh. *Provability Logics for Relative Interpretability*, Mathematical Logic, Springer, Boston, MA, 1990.
- G. Japaridze, D. de Jongh. *The Logic of Provability*, Handbook of Proof Theory, Elsevier, Amsterdam, 1998.

Theorem (Weak completeness)

If $\nvdash_{\mathrm{IL}} \varphi$, then there exists a finite Veltman model \mathfrak{M} such that $\mathfrak{M} \nvDash \varphi$.



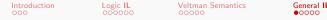
Completeness

Strong completeness, however, does not hold. Consider a set

$$S:=\{\Diamond p_0\}\cup\{p_n\to\Diamond p_{n+1}:n\in\mathbb{N}\}.$$

This set is consistent, because all of its finite subsets are consistent (we can find models which satisfy them). But the set S is not satisfied at any Veltman model. So, even though $S \nvDash_{IL} \perp$, there is no model which satisfies S and invalidates \perp .

Problem: S "needs" an infinite R-chain, which is impossible due to converse well-foundedness of R from the definition of Veltman frame.



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Quasi-Veltman models

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Definition

A quasi-Veltman frame \mathfrak{F} is a triple $(W, R, \{S_w : w \in W\})$, where W is a non-empty set, R transitive and irreflexive binary relation on W and $\{S_w : w \in W\}$ a collection of binary relations on R[w], where, for all $w \in W$, S_w is a reflexive and transitive and the restriction of R onto R[w] is contained in S_w .

Definition

A quasi-Veltmanov model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where \mathfrak{F} is a quasi-Veltman frame and $V : \operatorname{Prop} \to \mathcal{P}(W)$ is a valuation function.



Quasi-Veltman models

Completely analogous to Veltman model case, we define the forcing relation and we extend the valuation function to the set of all formulae.

Problem: There exist quasi-Veltman models which do not satisfy $\Box(\Box p \to p) \to \Box p$. Consider $(\mathbb{N}, <, \{ \leq |_{<[n]} : n \in \mathbb{N} \}, V)$, where $V(p) = 2\mathbb{N}$. Then

- for no $n \in \mathbb{N}$ does $n \Vdash \Box p$ hold,
- therefore, for all $n \in \mathbb{N}$, $n \Vdash \Box p \rightarrow p$ trivially holds,
- then $n \Vdash \Box (\Box p
 ightarrow p)$ obviously holds,
- finally: for no $n \in \mathbb{N}$ does $n \Vdash \Box(\Box p \to p) \to \Box p$ hold.

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Definition

A general IL-frame is a pair (\mathfrak{F}, A) , where \mathfrak{F} is a quasi-Veltman frame and $A \subseteq \mathcal{P}(W)$ a non-empty set of *admissible* subsets of W, closed under the following operations:

(i) *union:* if
$$X, Y \in A$$
, then $X \cup Y \in A$,

(ii) complement: if $X \in A$, then $W \setminus X \in A$,

(iii)
$$m_{arpi}$$
: if $X,Y\in A$, then $m_{arphi}(X,Y)\in A$, where

$$m_{\rhd}(X,Y) = \{w \in W : \forall u \in X (wRu \to \exists v \in Y (uS_wv))\},\$$

and satisfying the property

(iv) for all
$$X \in A$$
, $(W \setminus m_{\Box}((W \setminus m_{\Box}(X)) \cup X)) \cup m_{\Box}(X) = W$ holds.

We use $m_{\Box}(X)$ as a shorthand for $m_{\triangleright}(W \setminus X, \emptyset)$.



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Definition

A model based on a general IL-frame is a triple (\mathfrak{F}, A, V) , where (\mathfrak{F}, A) is a general IL-frame, and $V : \operatorname{Prop} \to A$ is an *admissible valuation*, which means that $V(p) \in A$, for all $p \in \operatorname{Prop}$.

Definition of a forcing relation is analogous to the one in the case of Veltman semantics, as is the extension of the valuation to the set of all formulas.



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We shall provide a sketch of the proof of the strong completeness, which is very similar to the proof of weak completeness for Veltman frames.

Definition

A set of formulae Γ is *consistent* if $\Gamma \nvDash_{\mathbf{IL}} \perp$. If Γ is a consistent set and, for any set Γ' , if $\Gamma \subsetneq \Gamma'$, then Γ' is inconsistent, then Γ is a *maximally consistent set*. Introduction 000 Veltman Semant

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Definition

Let Γ and Δ be two maximally consistent sets. We say that Δ is a *successor* of Γ , and denote it as $\Gamma \prec \Delta$, if

- for every formula $\Box \varphi \in \Gamma$, $\Box \varphi, \varphi \in \Delta$ holds and
- there exists a formula $\Box \psi \notin \Gamma$ such that $\Box \psi \in \Delta$.

If, additionally,

•
$$\neg \varphi, \Box \neg \varphi \in \Delta$$
 for all φ such that $\varphi \rhd \psi \in \Gamma$,

then Δ is a ψ -critical successor of Γ .



Assume that $\Gamma \nvdash_{\mathbf{IL}} \varphi$. Then $\Gamma \cup \{\neg \varphi\}$ is a consistent set. There exists a maximally consistent set Γ' which contains $\Gamma \cup \{\varphi\}$.

We define the following:

- W is the smallest set of pairs $w = (w_0, w_1)$, where w_0 is a maximally consistent set and w_1 a finite sequence of formulae such that
 - $(\Gamma', \langle \rangle) \in W$,
 - if $(w_0, w_1) \in W$, then $(w'_0, w_1) \in W$ i $(w'_0, w_1 * \langle \psi \rangle) \in W$ for each successor w'_0 of w_0 and formula ψ ,
- $wRv \iff w_0 \prec v_0 \text{ i } w_1 \subseteq v_1$,
- uS_wv if and only if:
 - $u, v \in R[w]$,
 - $w_1 = u_1 \subseteq v_1$, or $u_1 = w_1 * \langle \psi \rangle * \tau$ i $v_1 = w_1 * \langle \psi \rangle * \sigma$, for some formula ψ and finite sequences of formulae τ and σ , where, if u_0 is a ψ -critical successor of w_0 , then v_0 is, too.



We have defined a quasi-Veltman frame $\mathfrak{F} = (W, R, \{S_w : w \in W\})$. We define a valuation on this frame as follows:

$$w \in V(p) \iff p \in w_0.$$

By induction over the complexity of formulae, we can prove the following:

$$\mathbf{w}\Vdash\psi\iff\psi\in\mathbf{w_{0}},$$

for all $w \in W$ and formula ψ .

For this quasi-Veltman model $\mathfrak{M} = (\mathfrak{F}, V)$, both $\mathfrak{M}, (\Gamma', \langle \rangle) \Vdash \Gamma$ and $\mathfrak{M}, (\Gamma', \langle \rangle) \nvDash \varphi$ hold.



Finally, we define $A = \{V(\varphi) : \varphi \text{ formula}\}$, and we may directly check that $(W, R, \{S_w : w \in W\}, A)$ is a general **IL**-frame. Furthermore, V is admissible, therefore, the following theorem holds:

Theorem (Strong completeness)

Logic IL is sound and strongly complete with respect to the class of all general IL-frames.



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Properties of general **IL**-frames

- Every Veltman frame is a general IL-frame,
- If (ℑ, A, V) is a model based on a general IL-frame, then V(φ) ∈ A for all formulas φ,
- If (ℑ, V) is a quasi-Veltman model and if for a set A = {V(φ) : φ is a formula} the property (*iv*), holds, then (ℑ, A) is a general **IL**-frame.
- Logic IL is sound and strongly complete with respect to the class of all general IL-frames.



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This result

- improves upon Veltman semantics by giving us a frame-based semantics where strong completeness holds,
- generalizes the notion of general frame to the logic IL.



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Future work:

- classes of general frames for extensions of IL,
- general frames for Verbrugge semantics,
- algebraic semantics for interpretability logics.



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Thank you for your attention!

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