Are there mathematical concepts that are real?

Zvonimir Šikić

Carl Friedrich Gauss:

If $e^{i\pi} = -1$ *was not immediately apparent to a student upon being told it, that student would never become a first-class mathematician* [D].

Existence criterion

W. O. Quine [Q]: *Fs exist if* $\exists x F x$ *is a theorem of a true theory.*

M. Steiner [Mr]: *It is possible to satisfy this condition without being real.*

Reality criterion

P. Bridgman [B] p.56:
Something is physically real if it is connected with physical phenomena independent of those phenomena which entered its definition.

E. Nagel [N] p. 147:
A term designating anything physically real must enter into more than one experimental law, with the proviso that the laws are logically independent of each other.

I think that there is something profoundly right in the idea that the real is that which has properties transcending those which enter its definition.

To demonstrate the reality of an entity in the natural sciences one shows that the entity is indispensable in explaining some new phenomenon, independent of those phenomena which entered its definition. So, a real entity has a new and independent description.

Steiner applies the same idea in mathematics.

π is real because we have two independent descriptions for π.

1) Geometric, $\pi = C/2r$.
2) Analytic, $\pi = \ln(-1)/i$.

One is derived from $C = 2r\pi$, another from $e^{\pi i} = -1$.

In the natural sciences, we prove the coreference of two descriptions empirically. In mathematics, we do this with a deductive proof.

But when are provably coreferential descriptions independent?

Steiner's distinguish between proofs which are nonexplanatory and merely demonstrate the coreference, and those which explain it.

Descriptions are independent if the proofs of their coreferentiality are nonexplanatory.

The distinction is rather vague [Me]:

An explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property.

Steiner explains this difference only with a few examples.

Nonexplanatory proof that $\sqrt{2}$ is irrational:

If $a^2 = 2b^2$ and a/b reduced, then a^2 and hence a have to be even. Thus a^2 must be a multiple of 4, so b^2 and hence b must be multiples of 2 ($\perp$).

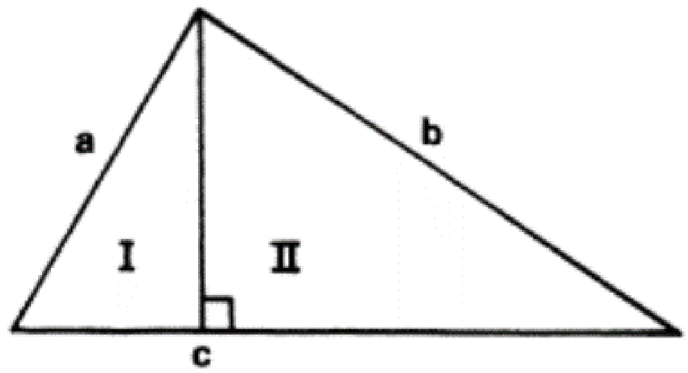
Nonexplanatory proof that $a^2 \neq nb^2$, unless n is a perfect square:

Assume a/b is reduced. If a prime p divides b and hence b^2, it must divide a^2 and hence a ($\perp$). So $b = 1$ and n is a perfect square.

Explanatory proof, because FTA is the characteristic property:

In the prime power expansion of a^2 the prime 2 will appear with an even exponent while in $2b^2$ its exponent must be odd. So a^2 never equals $2b^2$.

Steiner's example of the explanatory proof of the Pythagorean Theorem.



Three similar figures constructed on a, b, c have areas ka^2, kb^2, kc^2.

If $ka^2 + kb^2 = kc^2$, the Pythagorean Theorem follows immediately.

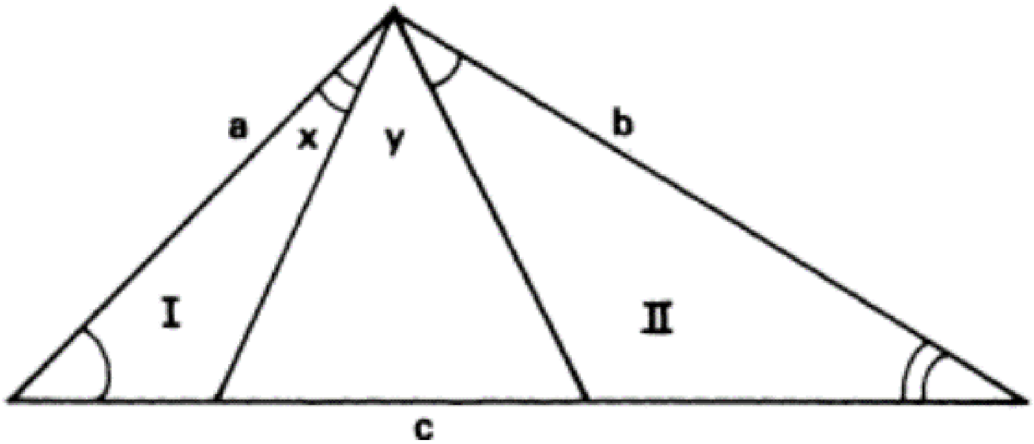
Thus Pythagorean Theorem is equivalent to a generalization; and the generalization to any of its instances.

Triangles I, II and T are similar to each other and I + II = T.

Steiner considers this proof to be explanatory because:

It characterizes the right triangle as the only one decomposable into two triangles similar to each other and to the whole.

The characterizing property for right triangles is the coincidence of x and y.



Nonexplanatory proof of $S(n) = \frac{n(n+1)}{2}$ by induction:

$$S(n+1) = S(n) + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

Explanatory proof, by symmetry:

1	+	2	+	3	+	...	+	n	=	S
n	+	(n-1)	+	(n-2)	+	...	+	1	=	S
(n+1)	+	(n+1)	+	(n+1)	+	...	+	(n+1)	=	S

The characterizing property is the symmetry of the sum.

It seems to me that the "*independence of the descriptions of two mathematical entities*" is not additionally explained by the "*absence of explanatory proofs of their coreference*", so I will stick to "*independence*".

As an example of real entity, Steiner tries to prove that π is real, because its geometric and analytic descriptions are independent [Mr].

To illustrate the claim, consider the astounding formula $\boldsymbol{e^{i\pi} + 1 = 0}$, *a "miracle" if anything is. It links the five fundamental constants of analysis in a most beautiful, simple way ... There seems no reason whatsoever to expect that such independently introduced numbers should be so simply related* [my emphasis].

According to Steiner, there seems no reason whatsoever to expect that π, as a constant of proportionality between the circumference and the radius of the circle, be in relation to e, i, 0 and 1, as claimed by Euler's formula, because the proof of the formula itself does not relate these two independent descriptions of π in any way.

Steiner is referring to a common proof:

Starting from the number e (= 2.72....), *defined as* $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$, ... *demonstrate using techniques of the calculus* [i.e. Taylor expansion of a function as an infinite polynomial], *that*

(1) $$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots .$$

In order to define complex exponentiation we substitute it for x

(2) $$e^{it} = 1 + \frac{it}{1!} - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \frac{it^5}{5!} - \cdots$$

$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) + i\left(\frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)$$

But it is known, using the same technique which yielded the expansion of e^x*, that for real t,*

(3)
$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$
$$\sin t = \frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

Substituting, we get the magic formula

(4)
$$e^{it} = cost + isint.$$

Substituting t = π, we have our results,

(5)
$$e^{i\pi} = -1.$$

Assuming that geometry and analysis are suitably independent, our criterion thus yields the reality of π.

Where did the magic come in?

Formula (1) is the standard Taylor's expansion.

Formula (2) is an extension of formula (1) to complex numbers.
I suppose this generalization is not questionable. It is a common way of extending definitions from the real to the complex field.

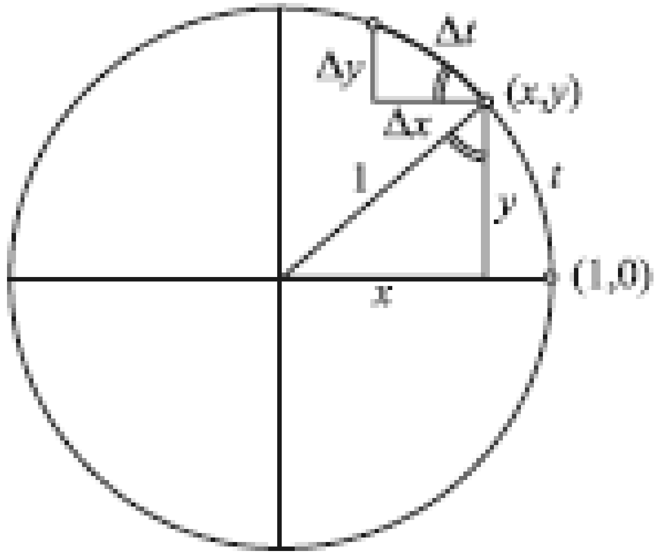
Transitions from (3) to (4) and (5) are trivial substitutions.

Hence, formula (3) remains as the only possible miracle:

How is it that the geometric descriptions of sinus and cosine coincide with the analytical descriptions in (3)?

There is no miracle here, cf. [Š].

Start with the geometric descriptions of $x = \sin t$ and $y = \cos t$:



It follows immediately that

$$\frac{dx}{dt} = -\frac{y}{1} \quad \& \quad \frac{dy}{dt} = \frac{x}{1} \qquad i.e. \qquad \frac{dcost}{dt} = -sint \quad \& \quad \frac{dsint}{dt} = cost$$

$$\int cost\, dt = sint + C \quad \& \quad \int sint\, dt = -cost + C$$

$$cosx \le 1 \Rightarrow \int_0^t cosx\,dx \le \int_0^t dx \Rightarrow sint \le t, for\ t > 0$$

$$sinx \le x, for\ x > 0$$

$$sinx \le x \Rightarrow \int_0^t sinx\,dx \le \int_0^t xdx \Rightarrow 1 - cost \le \frac{t^2}{2}, for\ t > 0$$

$$1 - x^2/2 \le cosx, for\ x > 0$$

$$1 - \frac{x^2}{2} \le cosx \Rightarrow \int_0^t \left(1 - \frac{x^2}{2}\right) dx \le \int_0^t cosxdx \Rightarrow t - \frac{t^3}{3!} \le sint, for\ t > 0$$

$$x - x^3/3! \le sinx, for\ x > 0$$

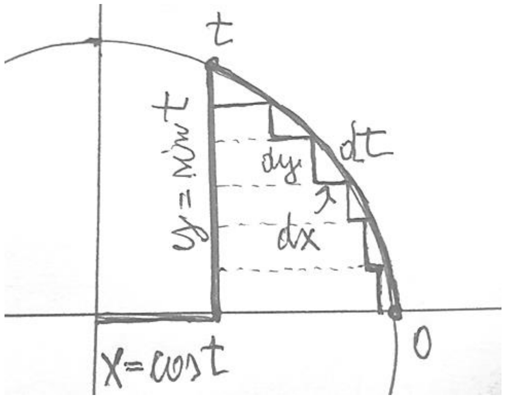
$$1-\frac{x^2}{2!}+\frac{x^4}{4}+\cdots+\frac{x^{4n-4}}{(4n-4)!}-\frac{x^{4n-2}}{(4n-2)!}\leq cosx\leq 1-\frac{x^2}{2!}+\frac{x^4}{4}+\cdots+\frac{x^{4n-4}}{(4n-4)!}$$

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{x^{4n-3}}{(4n-3)!}-\frac{x^{4n-1}}{(4n-1)!}\leq sinx\leq x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{x^{4n-3}}{(4n-3)!}$$

(Since the cosine is an even function and the sine is an odd function, it is obvious that the inequalities also hold for $x \leq 0$, i.e. they hold for all x.)

Hence, (3) is not independent of the geometric descriptions of sine and cosine. There are no miracles here.

And Newton, who first proved (3), started from the geometric description of sine and cosine [Nw]. There were no miracles even in the beginning.



$$x^2+y^2=1 \Rightarrow xdx+ydy=0 \Rightarrow (dx)^2=\frac{y^2(dy^2)}{x^2} \Rightarrow dt=\sqrt{dx^2+dy^2}=\frac{dy}{\sqrt{1-y^2}}$$

$$t=\int_0^t dt=\int_0^y \frac{dy}{\sqrt{1-y^2}}=(by\ Newton\ b.f.)=\int_0^y\left(1+\frac{1}{2}y^2-\frac{3}{4}y^4+\cdots\right)dy \Rightarrow$$

$$t=y+\frac{1}{6}y^3-\frac{3}{20}y^4+\cdots(by\ Newton\ method\ for\ solving\ equations) \Rightarrow$$

$$y=t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\frac{t^9}{9!}+\cdots$$

Let's deal with the very origin of the functions $y = ln\, x$ and $x = e^y$.

Many people are surprised by the fact that the first discovered logarithms were natural logarithms, that is, those with an unusual base 2.71828 … .

They are even more surprised when they learn that these logarithms are **the easiest to calculate** and that's why they were the first to be found.

Astonishment is justified if the logarithmic table is seen as a correspondence between real numbers x and y, because at the time of the discovery of logarithms, fractional powers were not known yet; exponents were exclusively natural numbers.

Nevertheless, natural logarithms were successfully defined for $x \in \mathbb{R}$?!

Tables of integer powers, like $x = 2^{\,Y}$, were known a long time ago.

Y	0	1	2	3	4	5	6	...
x	1	2	4	8	16	32	64	...

$$x_1 \cdot x_2 = x_3 \quad \Leftrightarrow \quad Y_1 + Y_2 = Y_3.$$

This clearly showed how multiplication can be reduced to addition.

However, the practical value of $x = 2^{\,Y}$ table is negligible due to the large differences between the available x's.

If the variable Y receive positive integer values, how will we get a finer distribution of x's? Take $b \approx 1$!

In 1614 Bürgi chose b = 1.0001 and Napier chose b = 0.9999999.

Let's follow Bürgi. In order to calculate $x = 1.0001^{\,Y+1}$, Bürgi used the previously calculated $x = 1.0001^{\,Y}$, but not for the calculation of the next x, but for the calculation of the increment to the next x.

$$x + \Delta x = 1.0001^{\,Y+1}$$

$$\Delta x = (x + \Delta x) - x \;=\; 1.0001^{\,Y+1} - 1.0001^{\,Y} \;=$$

$$1.0001^{\,Y}(1.0001 - 1) \;=\; \frac{x}{10000}.$$

How is the table of logarithms calculated? Very easy.

For given x, move the decimal point 4 places to the left, get Δx, add to x.

Y	x	Δx
0	1	0.0001
1	1.0001	0.00010001
2	1.00020001	0.000100020001
3	1.000300030001	Etc.

Scale down Y, by factor $\frac{1}{10^4}$, to get $y = \frac{Y}{10^4}$.

y	x
0.0000	1
0.0001	1.0001
0.0002	1.00020001
0.0003	1.000300030001

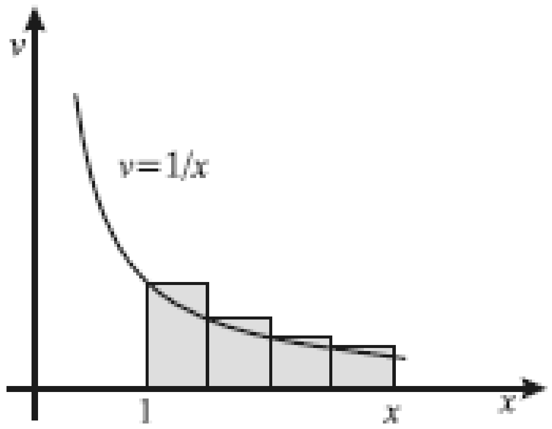
Let's see what is happening here from today's point of view. We have

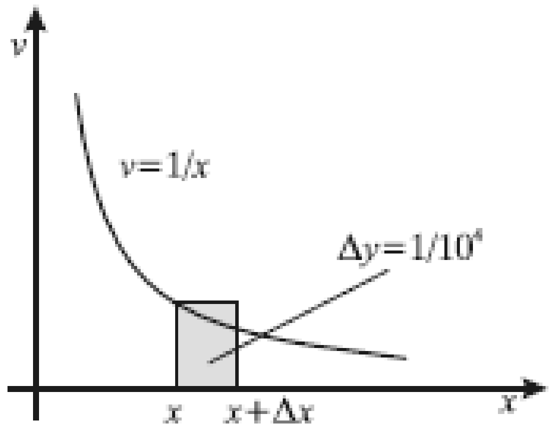
$$x = 1.0001^{Y} \;=\; 1.0001^{10000y} \;= \left(\left(1+\frac{1}{10000}\right)^{10000}\right)^{y} \;\approx e^{y} \text{ i.e. } y \approx ln\, x.$$

If, in the Y-table, we selected the base closer to 1 we would get finer distribution of x's and better approximation of natural logarithms.

But, the historical transition to natural logarithms was geometric.

$$\Delta y = \frac{1}{10^4} \;\&\; \Delta x = \frac{x}{10^4} \;\;\Rightarrow\;\; \Delta y = \frac{\Delta x}{x} \;\;\Rightarrow\;\; y = \sum_{1}^{x} \Delta y = \sum_{1}^{x} \frac{\Delta x}{x}$$





If the base b is closer to 1, then the rectangles have smaller areas, and at the limit we get the surface under the hyperbola, i.e.

$$\ln x = \int_1^x \frac{dx}{x}.$$

This transition was made by Mercator in 1667.

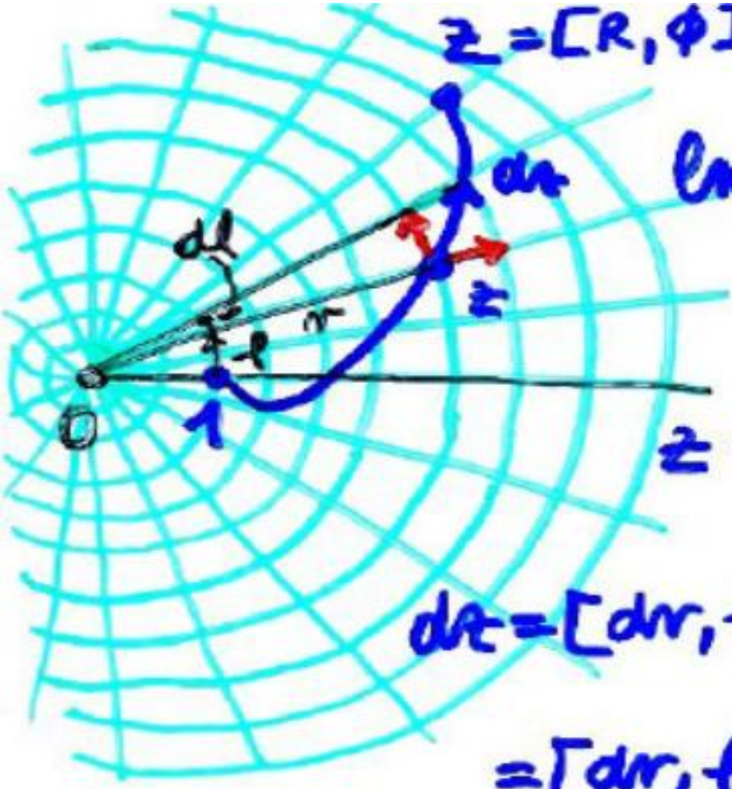
So, the natural definition of the natural logarithm in the complex plane is

$$\ln z = \int_1^z \frac{dz}{z}.$$

In $\mathbb{R}$, there is only one path from 1 to every z > 0.

In $\mathbb{C}$, there are many paths from 1 to every $z \neq 0$).

The calculation of $\ln Z$ is straightforward.



$$\ln z = \int_1^z \frac{dz}{z}$$

POLAR FORM

$$z = [r, l]$$

$$dz = [dr, l] + [r\,dl, l + \tfrac{\pi}{2}]$$

$$= [dr, l] + i[r\,dl, l]$$

$$\frac{dz}{z} = \frac{[dr, l]}{[r, l]} + i\,\frac{[r\,dl, l]}{[r, l]}$$

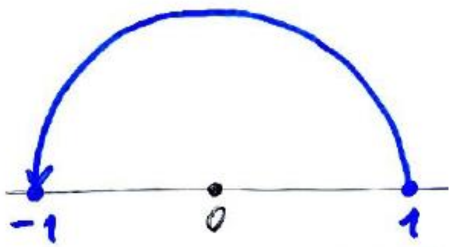
$$= \frac{dr}{r} + i\,dl$$

$$\boxed{\ln z = \int_1^R \frac{dr}{r} + i\int_0^\phi dl = \ln R + i(\phi + 2k\pi)}$$

Hence,

$$\ln(-1) = \int_1^1 \frac{dr}{r} + i\int_0^\pi d\varphi = i\pi.$$

It means that the analytic and geometric descriptions of **π** are identical:



$$\pi = \frac{\ln(-1)}{i} = \int_0^\pi d\varphi = \frac{C}{2} = \pi$$

In both cases it is the circumference of the semicircle.

We have proved beyond any doubt that π is not real in Steiner's sense.

As a matter of fact, it is difficult to prove for any mathematical concept that it is real in Steiner's sense.

We have to find two descriptions of a concept and a proof of their coreference, which keeps the descriptions independent.

But mathematical theories are deeply connected and mathematicians constantly strives to discover these connections.

For example, it is typical for mathematicians to persistently search for new proofs of old theorems in order to discover these dependencies.

My hypothesis is that no
mathematical concept
is real in Steiner's sense.

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Tycho Brahe′s assistent Paul Wittich introduced the method of prostaphairesis (προσθεσις = addition, αφαιρεσις = subtraction)

$$61620 \cdot 45318 = 2792495160\,?$$

$$0.61620 = \sin 38.039^\circ \qquad 0.45318 = \sin 26.948^\circ$$

$$\sin\alpha\sin\beta = (\cos(\alpha-\beta) - \cos(\alpha+\beta))/2$$

$$(\sin 38.039^\circ)(\sin 26.948^\circ) = (\cos 11.091^\circ - \cos\ 64.987^\circ)/2 =$$

$$(0.9813219765 - 0.42282294)/2 = 0.279249516$$