Logic and Applications

IUC Dubrovnik

September 23-27, 2024

Effective analogue of an ultraproduct of structures

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Standard models

- A *nonstandard* model of a theory is a model that is not isomorphic to the intended, standard model.

- For example, the *standard model* of Peano arithmetic, $\mathcal{N} = (\omega, +, \cdot, 0, 1)$, consists of the set $\omega = \{0, 1, 2, 3, \ldots\}$ with operations of addition $+$ and multiplication $\cdot$, and constants 0 and 1.

- The standard order of natural numbers is $\mathbb{N} = (\omega, <)$.

- *Complete number theory* is the set of all first-order sentences (with any number of quantifiers) which are true in the standard model of arithmetic.

Ultrapower construction

- Ultrapower constructions produce nonstandard models of theories.

- Let $\mathcal{A}$ be a countable structure with the domain A.

  An ultrafilter U is a certain set of large subsets of ω.

- An *ultrapower* is a direct product of countably infinitely many copies of $\mathcal{A}$ modulo $=_U$, in symbols $\mathcal{B} = \prod_U \mathcal{A}$, with the domain B.

- The elements of B are equivalence classes of infinite sequences f of elements in A : $(f(0), f(1), f(2), ...)$.

- The equivalence class of f is denoted by $[f]$.

  $[f] = [g]$ iff $\{i : f(i) = g(i)\} \in U$

- In $\prod_U \mathcal{N}$, $[\text{id}] = [(0, 1, 2, 3, \ldots)]$ is a new, nonstandard number.

- $\prod_U \mathcal{A}$ is typically uncountable. The existence of nontrivial U's uses Zorn's Lemma.

- $\prod_U \mathcal{A}$ has the same first-order theory as $\mathcal{A}$.

  We say they are *elementarily equivalent*.

Algorithmic (effective) ultrapower

- A structure is *computable.*

- Ultrafilters are replaced by infinite sets previously studied in computability theory, which are *indecomposable* with respect to *computably enumerable* sets.

- The elements of the product are equivalence classes of *partial computable functions*.

- Hence an effective ultrapower of a structure is a countable structure.

Computable structures

- A set is *computable* if there is a decision algorithm that recognizes its elements and non-elements.

- A countable structure $\mathcal{A}$ with finitely many operations and relations is *computable* if its domain is computable and its operations and relations are computable.

- *Examples of computable structures:* $\mathcal{N}$

  The ordered set of natural numbers, $\mathbb{N}$ (of order type ω)
  The ordered set integers, $\mathbb{Z}$
  The ordered set of rational numbers, $\mathbb{Q}$
  The additive group of integers, $(Z, +, 0)$
  The field of rational numbers, $(Q, +, \cdot, 0, 1)$

- *Example of a non-computable structure*

  Let H be a non-computable set, say the halting set.

  Define a linear order $(\{0, 1, 2, \ldots\}, \prec)$ isomorphic to $\mathbb{N}$ (of order type ω):

  $2n \prec 2n + 1$ if $n \in H$
  $2n + 1 \prec 2n$ if $n \notin H$

  $2n,\ 2n + 1 \prec 2n + 2, 2n + 3$

  $0, 1 \prec 2, 3 \prec 4, 5 \prec \cdots$

- If this order were computable, then H would be computable.

- Tennenbaum's Theorem
  There is no computable nonstandard model of Peano arithmetic.

Computably enumerable sets

- A nonempty set W of natural numbers is *computably enumerable* if there is an algorithm that generates it by enumerating (i.e., listing) its elements: $W = \{w_0, w_1, w_2, \ldots\}$

- If W is finite or its elements can be algorithmically enumerated in strictly increasing order, then W is *computable*.

- There are many non-computable computably enumerable sets.

Partial computable functions

- Let $P_0, P_1, ..., P_e, ...$ be an algorithmic enumeration (given by systematic listing) of all Turing machine programs.

- Turing machine program P_e computes a *partial computable* (possibly total, thus computable) function φ_e:

  on input x, it halts and outputs its value, in symbols $\varphi_e(x)\downarrow$, when $x \in dom(\varphi_e)$, or it computes forever, in symbols $\varphi_e(x)\uparrow$, when $x \notin dom(\varphi_e)$.

- Also, we have an algorithmic enumeration of all computably enumerable sets:

  $W_0, W_1, ..., W_e, ...$

Cohesive Sets

- A set C of natural numbers is *cohesive* if C is infinite and for every computably enumerable set W, either $W \cap C$ or $\overline{W} \cap C$ is finite.

  Hence:

  $W \cap C$ is infinite $\Rightarrow C \subseteq^* W$
  $\overline{W} \cap C$ is infinite $\Rightarrow C \subseteq^* \overline{W}$

  $\subseteq^*$ stands for inclusion of all but finitely many elements

- Every infinite set of natural numbers has a cohesive subset.

Effective (Cohesive) Ultrapowers

- Let $\mathcal{A}$ be a computable structure with domain A, and let C be a cohesive set of natural numbers.

  The *cohesive ultrapower* of $\mathcal{A}$ over C, in symbols $\mathcal{B} = \Pi_C \mathcal{A}$, has the domain ($D$ mod $=_C$) where

  $D = \{\varphi \mid \varphi : \omega \to A$ is partial computable and $C \subseteq^* dom(\varphi)\}$.

  For $\varphi, \psi \in D$, define

  $$\varphi =_C \psi \quad \text{iff} \quad C \subseteq^* \{i : \varphi(i) \downarrow = \psi(i) \downarrow\}.$$

  The equivalence class of φ is denoted by $[\varphi]$.

- If F is an n-ary operation (function) symbol, then

$$F^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n]) = [\varphi],$$

  where for every $i \in \omega$,

$$\varphi(i) = F^{\mathcal{A}}(\varphi_1(i), \ldots, \varphi_n(i)),$$

  equal as partial functions.

- If R is an m-ary relation symbol, then

$$R^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_m]) \quad \text{iff} \quad C \subseteq^* \{i \in \omega : R^{\mathcal{A}}(\varphi_1(i), \ldots, \varphi_m(i))\}$$

- If c is a constant symbol, then $c^{\mathcal{B}}$ is the equivalence class of the computable function with constant value $c^{\mathcal{A}}$.

- Canonical embedding of $\mathcal{A}$ into $\Pi_C\mathcal{A}$: $a \to [\theta_a]$, where $\theta_a = (a, a, ...)$.

- For a finite structure $\mathcal{A}$, we have $\Pi_C\mathcal{A} \cong \mathcal{A}$.

- For an infinite computable structure $\mathcal{A}$, the effective ultrapower $\Pi_C\mathcal{A}$ and $\mathcal{A}$ do not necessarily have the same theory.

- If $\mathcal{A}$ and $\mathcal{B}$ are *computably isomorphic*, then $\Pi_C\mathcal{A} \cong \Pi_C\mathcal{B}$.

Preservation of satisfaction

- Dimitrov's Theorem

  (i) If σ is a $\forall\exists$ (or $\exists\forall$) sentence, then

  $$\Pi_C \mathcal{A} \models \sigma \quad \text{iff} \quad \mathcal{A} \models \sigma$$

  (ii) If σ is a $\exists\forall\exists$ sentence, then

  $$\text{if } \mathcal{A} \models \sigma \quad \text{then} \quad \Pi_C \mathcal{A} \models \sigma$$

- If $\mathcal{A}$ has more decidability, then more satisfaction is preserved.

- If a computable structure $\mathcal{A}$ is from one of the following classes, then so is its effective ultrapower $\Pi_C \mathcal{A}$:
  - fields
  - structures with an equivalence relation
  - graphs
  - structures with a single one-to-one function (directed graphs)
  - other directed graphs obtained from functions
  - linear orders

- There are $\forall\exists\forall$ sentences true in some computable $\mathcal{A}$, but not in $\Pi_C \mathcal{A}$ (for some C).

- (Feferman, Scott and Tennenbaum; Lerman)

  There is a $\forall\exists\forall$ sentence (involving Kleene's T predicate), which is true in $\mathcal{N}$, the standard model of arithmetic, but not in $\Pi_C \mathcal{N}$.

- **Proof sketch.**

  Let P_e be the e-th Turing machine program.

  In Kleene's predicate $T(e, x, z)$, e refers to P_e, x is the input, and z codes the output and the number s of computation steps.

  Consider the sentence:

  $(\forall x)(\exists s)(\forall e \leq x)[P_e(x) \downarrow \Rightarrow P_{e, \leq s}(x) \downarrow]$

Cohesive powers and their isomorphism types have been studied for:

- The field of rational numbers, $(Q, +, \cdot, 0, 1)$, by Dimitrov, Harizanov, R. Miller and J. Mourad

- Linear orders by Dimitrov, Harizanov, Morozov, Shafer, A. Soskova and Vatev

- Structures with an equivalence relation, certain graphs, and function structures (A, f) for various unary functions by Harizanov and Srinivasan

Cohesive powers of linear orders

- We use $+$ for the sum and $\times$ for the lexicographical product of two linear orders.

- We can show that for $\mathbb{N}$, we have $\Pi_C \mathbb{N} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.

- Let $\mathcal{L}$ be a computable dense linear order without endpoints, say $\mathbb{Q}$. Then $\Pi_C \mathcal{L} \cong \mathcal{L}$.

- **Proof**. The first-order theory of dense linear orders without endpoints is $\forall\exists$-axiomatizable and countably categorical (has only one countable model, up to isomorphism).

  $\Pi_C \mathcal{L}$ is countable, so $\Pi_C \mathcal{L} \cong \mathcal{L}$.

- Assume that $\mathcal{L}, \mathcal{L}_0, \mathcal{L}_1$ are computable linear orders, and $\mathcal{L}^{rev}$ is the reverse of $\mathcal{L}$.

- $\Pi_C(\mathcal{L}_0 + \mathcal{L}_1) \cong \Pi_C\mathcal{L}_0 + \Pi_C\mathcal{L}_1$

  $\Pi_C(\mathcal{L}_0 \times \mathcal{L}_1) \cong \Pi_C\mathcal{L}_0 \times \Pi_C\mathcal{L}_1$

  $\Pi_C\mathcal{L}^{rev} \cong (\Pi_C\mathcal{L})^{rev}$

- For example,

  $\Pi_C\mathbb{N}^{rev} \cong (\Pi_C\mathbb{N})^{rev} \cong (\mathbb{N} + (\mathbb{Q} \times \mathbb{Z}))^{rev} \cong (\mathbb{Q} \times \mathbb{Z}) + \mathbb{N}^{rev}$

- Similarly,

  $\Pi_C\mathbb{Z} \cong \Pi_C(\mathbb{N}^{rev} + \mathbb{N}) \cong \mathbb{Q} \times \mathbb{Z}$

When the successor function is computable

- Let $\mathcal{A}$ be a computable linear order of order type ω, with a *computable* successor function. Then for every cohesive set C, we have $\Pi_C \mathcal{A} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.

  $\mathcal{A}$ is computably isomorphic to the standard model $\mathbb{N}$.

- Having a computable successor function is not necessary for this order type of an effective ultrapower.

- There is a computable linear order $\mathcal{A}$ of order type ω, with a *non-computable* successor function, such that for every cohesive C, we have $\Pi_C \mathcal{A} \cong \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$.

When $\Pi_C \mathcal{L} \not\equiv \mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$

- Let C be a cohesive set. There is a computable linear order $\mathcal{L}$ of order type ω such $\Pi_C \mathcal{L}$ and $\mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ are not elementarily equivalent.

- **Proof sketch**

  Construct a computable linear order $\mathcal{L} = (X, <_{\mathcal{L}})$ of order type ω.

  Assure that if φ is a partial computable function such that

  [id] $<_{\Pi_C \mathcal{L}}$ [φ], then [φ] is not the $<_{\Pi_C \mathcal{L}}$-immediate successor of [id].

- It follows that $\Pi_C\mathcal{L}$ and $\mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ are not elementarily equivalent because every element of $\mathbb{N} + (\mathbb{Q} \times \mathbb{Z})$ has an immediate successor, but $[\mathrm{id}] \in \Pi_C\mathcal{L}$ does not have an immediate successor.

- The sentence σ that states that every element has an immediate successor is a $\forall\exists\forall$-sentence. Then for the computable linear order $\mathcal{L}$ of type ω, constructed above, we have $\mathcal{L} \models \sigma$ but $\Pi_C\mathcal{L} \models \neg\sigma$.

When cohesive sets have computably enumerable complements

- A set $M \subseteq \omega$ is *maximal* if M is computably enumerable and its complement $\overline{M} = C$ is cohesive.

  Equivalently, M is computably enumerable, $\overline{M}$ is infinite, and for every computably enumerable set W with $M \subseteq W \subseteq \omega$, either $\omega - W$ or $W - M$ is finite.

- For every $[\varphi] \in \Pi_C \mathcal{A}$, there is a (total) computable function f such that $[f] = [\varphi]$.

- **Proof**. Define $\widehat{f}(n) = \begin{cases} \varphi(n) & \text{if } \varphi(n)\downarrow \text{ first,} \\ 0 & \text{if } n \text{ is enumerated into } M \text{ first.} \end{cases}$

  $\widehat{f}(n)$ is defined for all but finitely many n.

- Fix C to be a co-maximal set.

- There is a computable linear order $\mathcal{L}$ of order type ω such that $\Pi_C \mathcal{L} \cong \mathbb{N} + \mathbb{Q}$.

- There is a countable set of computable linear orders of order type ω, the effective ultrapowers of which are pairwise non-elementarily equivalent.

- It is possible for non-elementarily equivalent computable linear orders to have isomorphic effective ultrapowers.

- Let $\mathcal{X}$ be a non-empty, at most countable set of order types.

  Let $|\mathcal{X}|$ be the size of $\mathcal{X}$.

- The shuffle $sh(\mathcal{X})$ is obtained by densely coloring $\mathbb{Q}$ with $|\mathcal{X}|$ many colors, assigning to each order type in $\mathcal{X}$ a distinct color and replacing each $q \in \mathbb{Q}$ with the order type corresponding to the color of q.

- Let C be a co-maximal set.

- Let $k_0, ..., k_n$ be positive natural numbers, and $\mathbf{k}_0, ..., \mathbf{k}_n$ the corresponding ordered sets.

  $\mathbf{k}$ is $0 < 1 < \cdots < k - 1$

- There is a computable linear order $\mathcal{M}$ of order type ω such that $\Pi_C\mathcal{M}$ has order type $\omega + sh(\mathbf{k}_0, ..., \mathbf{k}_n)$.

- Let $\mathcal{X}$ be a Π^0_2 or Σ^0_2 (possibly infinite) set of finite non-empty order types. Then there is a computable linear order $\mathcal{L}$ of order type ω such that $\Pi_C\mathcal{L}$ has order type $\omega + sh(\mathcal{X} \cup \{\mathbb{N} + (\mathbb{Q} \times \mathbb{Z}) + \mathbb{N}^{rev}\})$.

THANK YOU!

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[4] V. Harizanov and K. Srinivasan, *Archive for Mathematical Logic* 63 (2024), pp. 679–702.

(joint work with Rumen Dimitrov, Andrei Morozov, Paul Shafer, Alexandra Soskova and Stefan Vatev)

- **Fundamental Theorem for $\Pi_C \mathcal{A}_i$.**

  Let $(\mathcal{A}_i)_{i \in \omega}$ be a sequence of uniformly n-decidable structures, and let C be a cohesive set.

  (1) Let $\alpha(x_1, \ldots, x_m)$ be a Σ^0_{n+2} formula. Then

  $\Pi_C \mathcal{A}_i \models \alpha([\varphi_1], \ldots, [\varphi_m]) \;\Rightarrow\; C \subseteq^* \{i : \mathcal{A}_i \models \alpha(\varphi_1(i), \ldots, \varphi_m(i))\}$

  (2) The converse holds for a Π^0_{n+2} formula.

  (3) The equivalence holds for a Δ^0_{n+2} formula.

- Say that a formula is Δ^0_k if it is logically equivalent to both a Σ^0_k formula and a Π^0_k formula.

- If $\mathcal{A}$ is a decidable structure, then $\mathcal{A}$ and $\Pi_C\mathcal{A}$ are elementarily equivalent.

- Let $\mathcal{A}$ be an n-decidable structure.

  Then $\mathcal{A}$ and $\Pi_C\mathcal{A}$ satisfy the same Δ^0_{n+3} sentences.

  If σ is a Σ^0_{n+3} sentence, then $\mathcal{A} \models \sigma \Rightarrow \Pi_C\mathcal{A} \models \sigma$.