# Computable approximations of semicomputable graphs

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# Brief overview of the background

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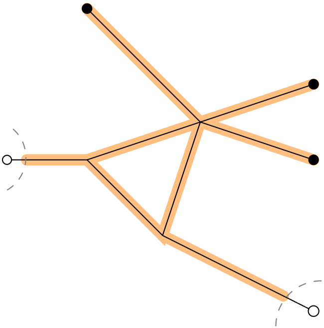
- Typically, when speaking of semicomputable sets, we restrict ourselves to compact sets.
- However, fairly tame non-compact sets are also fine (e.g. sets whose intersection with each closed ball is compact).

# Our goal and the plan

- We consider *(generalized topological) graphs*, i.e. disjoint unions of finitely many arcs and rays, where these arcs and rays may be glued together at their endpoints.
- Goal: prove that semicomputable graphs can be approximated by computable subgraphs with computable endpoints.
- The plan is to simply nip the graph a(rbitrarily) little at its uncomputable endpoints while preserving semicomputability.



- Our goal is then immediately achieved...

# The known result and our improvements

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Theorem (Iljazović, 2020).

$(X, d, \alpha)$ computable metric space, $S \subseteq X$ semicomputable graph, set of all endpoints of $S$ semicomputable. Then $S$ is computable.

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- We also cover generalized graphs and uncomputable endpoints

**Theorem (ČČHI, 2024).**

$(X, d, \alpha)$ computable metric space, $S \subseteq X$ semicomputable generalized graph, $\varepsilon > 0$. Then there exists a computable generalized graph $T$ in $(X, d, \alpha)$ such that $T \subseteq S$, all endpoints of $T$ are computable and $d_H(S, T) < \varepsilon$.

$d_H(S, T) = \inf\{\varepsilon > 0 : S \approx_\varepsilon T\}$

$$S \approx_\varepsilon T \iff (\forall x \in S)(\exists y \in T)(d(x,y) < \varepsilon) \text{ and } (\forall x \in T)(\exists y \in S)(d(x,y) < \varepsilon)$$

# The nipping

- How do we actually perform the nipping? First, we used the following...

Theorem (Iljazović, Validžić, 2017).

$S$ semicomputable in $(X, d, \alpha)$, $x \in S$ has a neighbourhood homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N} \setminus \{0\}$. Then $x$ has a computable compact neighbourhood in $S$.

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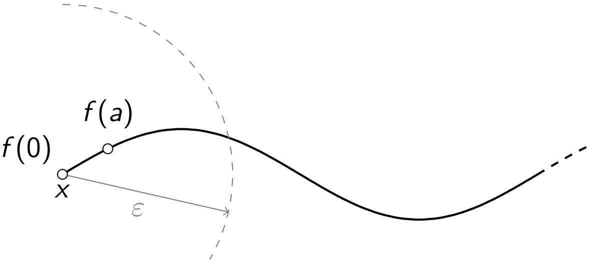
- ...to prove a lemma:

**Lemma.**
$S$ semicomputable in $(X, d, \alpha)$, $x \in S$ has a neighborhood $N$ in $S$ which is homeomorphic to $\mathbb{R}$. Then there exist computable points $a, b \in N$ and a computable neighbourhood $N' \subseteq N$ of $x$ in $S$ which is an arc from $a$ to $b$.

# The nipping

### Lemma (Nipping).

$S$ semicomputable in $(X, d, \alpha)$, $x \in S$ has an open neighborhood $N$ in $S$ such that there exists a homeomorphism where $f\colon [0,1\rangle \to N$ and $f(0) = x$, $\varepsilon > 0$. Then there exists $a \in \langle 0,1\rangle$ such that $f(a)$ is a computable point, $S \setminus f([0,a\rangle)$ is semicomputable and $f([0,a]) \subseteq B(x,\varepsilon)$.



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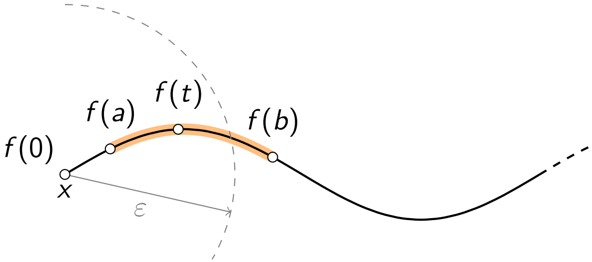


$f$ continuous $\Rightarrow \exists t \in \langle 0, 1\rangle.\ f([0, t]) \subseteq B(x, \varepsilon)$
$f(\langle 0, 1\rangle)$ open in $N$ (so in $S$ too) $\Rightarrow f(\langle 0, 1\rangle)$ open neighborhood of $f(t)$ in $S$, homeomorphic to $\mathbb{R} \overset{\text{Lemma}}{\Longrightarrow} \exists$ computable neighborhood $N'$ of $f(t)$ in $S$, $N' \subseteq f(\langle 0, 1\rangle)$, $N'$ arc with computable endpoints.
Since $f$ is a homeomorphism, $f^{-1}(N')$ is an arc in $\langle 0, 1\rangle$, thus it is equal to $[a, b]$ for some $a, b \in \langle 0, 1\rangle$, $a < t < b$.
Clearly, $f(a)$ is a computable point (as an endpoint of $N'$) and

$$f([0, a]) \subseteq f([0, t]) \subseteq B(x, \varepsilon).$$

All that remains: $S \setminus f([0, a\rangle)$ is semicomputable.

# The nipping

Why is $S \setminus f([0,a\rangle)$ semicomputable? First, $[0,b\rangle$ is open in $[0,1\rangle$, so $f([0,b\rangle)$ is open in N, and thus in $S$. Let $U \subseteq X$ open such that $f([0,b\rangle) = S \cap U$. $f([0,a])$ is a compact subset of $U$, so there exists $m \in \mathbb{N}$ such that $f([0,a]) \subseteq J_m \subseteq U$ $(*)$.

,,We" prove that $S \setminus J_m$ is semicomputable. First, for a closed ball $B$, $(S \setminus J_m) \cap B$ is compact (it is closed and contained in the compact set $S \cap B$). Second, let $\Omega_A = \{(i,j) \in \mathbb{N}^2 \mid \hat{I}_i \cap A \subseteq J_j\}$ and $\phi : \mathbb{N}^2 \to \mathbb{N}$ computable, $J_a \cup J_b = J_{\phi(a,b)}$ for $a, b \in \mathbb{N}$. Then

$$(i,j) \in \Omega_{S \setminus J_m} \Leftrightarrow \hat{I}_i \cap S \subseteq J_j \cup J_m = J_{\phi(j,m)} \Leftrightarrow (i, \phi(j,m)) \in \Omega_S.$$

Hence, $(S \setminus J_m) \cup f([a,b])$ is semicomputable (so if we prove it is equal to $S \setminus f([0,a\rangle)$, we are done).

$\boxed{\subseteq}$ $S \setminus J_m \subseteq S \setminus f([0,a])$ from $(*)$; $f$ inj.$\Rightarrow$ $f([a,b]) \cap f([0,a\rangle) = \emptyset$
$\boxed{\supseteq}$ for $y \in S \setminus f([0,a\rangle)$, suppose $y \in J_m$; then by $(*)$, $y \in U \cap S = f([0,b\rangle) = f([0,a\rangle) \cup f([a,b\rangle)$, so $y \in f([a,b\rangle)$ $\square$

# Nice bonus: 1-manifolds with boundary

- 1-manifold with boundary: second countable Hausdorff space $+$ each point has a neighborhood homeomorphic to $[0, \infty\rangle$
- boundary $\partial M$: set of all $x \in M$ such that $x$ has a neighborhood in $M$ homeomorphic to $[0, \infty\rangle$ by a homeomorphism which maps $x$ to $0$
- Each component is known to be homeomorphic to:

$$\mathbb{R}, \quad [0, \infty\rangle, \quad \mathbb{S}^1, \quad \text{or} \quad [0, 1].$$

  - $\mathbb{R}$: (homeomorphic to) two rays welded together
  - $[0, \infty\rangle$: a single ray
  - $\mathbb{S}^1$: two arcs welded together
  - $[0, 1]$: arc
- Conclusion: if a 1-manifold with boundary has finitely many components, it is a generalized graph with endpoints $\partial M$.

# Nice bonus: 1-manifolds with boundary

### Corollary

Let $(X, d, \alpha)$ be a computable metric space and let $M$ be a semicomputable 1-manifold in this space such that $M$ has finitely many components. Then for each $\varepsilon > 0$ there exists a computable 1-manifold $N$ in $(X, d, \alpha)$ such that $N \subseteq M$, each point of $\partial N$ is computable and $M \approx_\varepsilon N$.