Brouwer-Weyl Continuum Through 3D Glasses:

Geometry, Computation, General Relativity

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Motivation?

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I happened to be writing an article on Ian Hacking on the 19th century confusions regarding the concept of determinism when I came across the sentence above, in Biggs & Tucker, 'Can Newtonian systems, bounded in space, time, mass and energy compute all functions?' (*Theoretical Computer Science* 371, 2007).

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If we are not even sure what counts as a Newtonian system, what should we make of the growing literature on the question of whether physical computational devices can violate the Church-Turing Thesis?

Example: Zeno-Hilbert Hotel

In the 1990s Pérez Laraudogoitia presented an elementary example of failure of determinism in Newtonian mechanics.

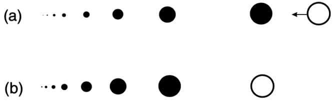

Figure 1.1
Zeno's revenge.

Image from John Earman, 'Determinism: What We Have Learned and What We Still Don't Know' (2004).

There are far more complicated examples, sometimes related to solutions of long-standing problems such as the Painlevé Conjecture. It is also possible to give such examples in special relativity and with finite total mass of the system.

Example: Norton's Dome

Another simple example was devised by John Norton in 2008. Here a point mass is placed at the apex of a circular dome.

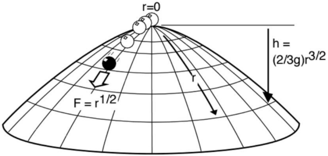

Image from John Norton, 'The Dome: An Unexpectedly Simple Failure of Determinism' (2008).

The equation of motion under gravity is given by

$$\frac{d^2 r}{dt^2} = \sqrt{r} \qquad r(0) = 0 \qquad r'(0) = 0$$

where $r(t)$ is the distance from the apex of the dome. Expected solution is $r = 0$. But solutions are not unique. Apparently the mass can start rolling down the dome at an arbitrary time, in any direction.

A bit of confusion

On the one hand, we have a well known mathematical physicist (Malament) saying — in 2008 — we're not sure what Newtonian systems are supposed to be. And *The Oxford Handbook of Philosophy of Physics* (2013) included a substantial article on 'What is "Classical Mechanics" Anyway?'

On the other hand, there are arguments that devices based on classical mechanics are capable of going beyond the Church-Turing Thesis. E.g., Beggs & Tucker (*Theoretical Computer Science*, 2007) write: 'simple Newtonian kinematic systems that are bounded in space, time, mass and energy can compute all possible sets and functions on discrete data.'

A more specific confusion

In a seminal paper from the 1980s, Pour-El and Richards showed that the standard 3D wave equation can have a computable initial condition at $t = 0$ but the solution is uncomputable at $t = 1$. ('The Wave Equation with Computable Initial Data Such That Its Unique Solution Is Not Computable', *Advances in Mathematics*, 1981).

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Two reactions in the mathematical physics community:

- This is crazy! We routinely compute solutions of the wave equation!
- This is awesome! We can now imagine a 'wave computer', a physical computing system that violates the Church-Turing Thesis!

On the other hand…

Some twenty years later, Weihrauch and Zhong analyzed the Pour-El-Richards result using the finer framework of Type-2 Theory of Effectivity. They conclude:

> *even under very idealizing assumptions about measurements and wave propagation in reality, it seems to be very unlikely that the Pour-El-Richards counterexample can be used to build a physical machine with a 'wave subroutine' computing a function which is not Turing computable. We may still believe that the Church-Turing Thesis holds.*

('Is Wave Propagation Computable or Can Wave Computers Beat the Turing Machine?', *Proc. London Math. Soc.* 2002)

Enter General Relativity

Malament-Hogarth spacetime makes it possible for a Turing machine to travel along a trajectory that has infinite proper time and send a signal to an observer in whose frame the machine's trajectory has finite time. In this setting, the observer would have at their disposal an infinite-time Turing machine. Some uncomputable functions would be computable in a such a world.

Combining these spacetimes ultimately leads to the conclusion that arithmetic would be decidable in a (specially designed) relativistic spacetime. And more than arithmetic, as shown by Welch ('The Extent of Computation in Malament-Hogarth Spacetimes', 2018).

More 'realistic' spacetimes

The discussion of Malament-Hogarth spacetimes usually centres on whether they are physically reasonable. Usual answer is no, but this relies on a certain hypothesis on what is a 'physically reasonable' spacetime; which has been debated.

Supposedly more realistic models, e.g. Kerr rotating black hole cosmology, have been utilized to argue the possibility of violations of the Church-Turing Thesis. This argument was proposed by Etesi and Németi in 'Non-Turing computations via Malament-Hogarth specetimes' (*Int. J. Theoretical Physics*, 2002).

A radical claim, a metaphor from history geometry

In his 'Non-Turing Computers are the New Non-Euclidean Geometries' (*International Journal of Unconventional Computation*, 2009), Hogarth — of Hogarth-Malament spacetimes — formulates the idea that the concept of computability is dependent on the physical context in which a 'machine' resides:

> *non-Turing computers are viewed as one views non-Euclidean geometries. [...] To put it another way, the Church-Turing Thesis is like the outmoded claim: "Euclidean geometry is the true geometry."*

But the 'physical' spacetimes in which these non-standard computations purportedly occur are, of course, mathematical models of an assumed ambient physics.

Brouwer-Weyl continnum in 3D

Let’s not start with physics but with a mathematical concept related (loosely) to computation, and then see whether some physics can be made to enter the picture. Why not.

According to Brouwer (and, for a time, Weyl), the continuum should be regarded as the collection of ‘sequences of nested intervals whose measure converges to zero.’

A higher-dimensional analog would be nested sequences of spheres with radii converging to zero.

Let’s not worry about centres and radii being rational, we’ll come back to that.

Some classical geometry

Classical geometry, going back to Laguerre and Lie, encodes the space of *oriented spheres* in $\mathbb{R}^n$ as points in $\mathbb{R}^{n+1}$: $(\mathbf{x}, r)$ with $\mathbf{x} \in \mathbb{R}^n$ being the centre and $r \in \mathbb{R}$ the oriented radius. In this cyclographic representation the space of spheres has the structure of the Minkowski space $\mathbb{R}^{1,n}$ with the usual pseudometric.

Then for $r_1, r_2 > 0$, $||\mathbf{x}_1 - \mathbf{x}_2|| \leq r_1 - r_2$ iff the sphere $(\mathbf{x}_2, r_2)$ is contained in the sphere $(\mathbf{x}_1, r_1)$

In the terminology of special relativity, sphere inclusion corresponds to events that are related in the causal order.

The concept of a nested sequence of spheres thus corresponds to a time-oriented causal sequence of events in Minkowski space.

Restricting to positive radii

Laguerre and Lie were interested in *oriented* spheres. In the Brouwer-Weyl approach there is no mention of orientation of intervals, so we should restrict to positive radii.

Let's look at a different representation of the space of spheres, in terms of Lie cycles. For a sphere $(\mathbf{x}, r)$, with $r > 0$, consider the vector $(y_0, \ldots, y_{n+1}) \in \mathbb{R}^{1,n+2}$ given by

$$\begin{aligned} y_0 &= -\frac{1}{2}\Big(\frac{||\mathbf{x}||^2 + 1}{r} - r\Big) \\ (y_1, \ldots, y_n) &= -\frac{1}{r}\mathbf{x} \\ y_{n+1} &= -\frac{1}{2}\Big(\frac{||\mathbf{x}||^2 - 1}{r} - r\Big) \end{aligned}$$

deSitter spacetime as space of spheres

With the coordinate transformations on the previous slide, we have $-y_0^2 + \sum_{k=1}^{n+1} y_k^2 = 1$. Inducing the metric on this hyperboloid from $\mathbb{R}^{1,n+2}$, one gets the deSitter metric on the space of spheres

$$ds^2 = \frac{1}{r^2}(-dr^2 + d\mathbf{x}^2).$$

Substitution $r = e^{\mp t}$ yields the usual form of the deSitter metric

$$ds^2 = -dt^2 + e^{\pm 2t} d\mathbf{x}^2$$

in flat slicing coordinates of the "expanding" (resp. "contracting") part.

Brouwer-Weyl continuum as space of causal curves

Our detour through classical geometries relates the "higher-dimensional continuum" to a well known object in general relativity.

Nested sequences of spheres correspond to "time"-oriented causal sequences.

But without additional qualifications, such sequences could be finite. A more precise analog would be:

Nested sequences of spheres in the 3D analog of the Brouwer-Weyl continuum correspond to inextendible "time"-oriented causal sequences (by analogy of inextendible causal curves): there is no sphere that is contained in all spheres in the nested sequence.

One might think of them as computations that cannot be made more precise?

Where does all this come from?

Order theorists are interested in inclusion representations of posets. That is, one takes a set $\mathcal{A}$ of subsets of some space (e.g. spheres in $\mathbb{R}^n$, or intervals) and considers the set-theoretic inclusion relation among them. The question then arises of which posets admit an inclusion representation using objects in $\mathcal{A}$.

In mathematical physics there is the Causal Set program, which seeks to derive spacetime structure from discrete posets. The question then arises of which posets can plausibly occur as part of the causal order of a spacetime. This problem, at least in the case of deSitter spacetime, relates to inclusion representation of posets by spheres.

Can this be made discrete, in some sense?

Preceding discussion has a fatal flaw in that it relies on classical analysis. Let's see if we can work around it.

Martin and Panangaden ('Domain Theory and General Relativity', in *New Structures for Physics*, 2010) introduce the category of globally hyperbolic posets, which includes causal orders on globally hyperbolic spacetimes such as deSitter.

This category is equivalent to the category of interval domains, introduced by Scott in his pioneering work on the theory of computation and semantics of programming languages.

Domain Theory aspect

The upshot of the argument by Martin and Panangaden is that manifold topology (if not geometry) of a globally hyperbolic spacetime — such as deSitter — can be recovered from a countable dense subset of the associated interval domain of the causal order.

The spacetime itself (if we start from one) is homeomorphic to the set of maximal elements in the interval domain, with Scott topology.

If no manifold is given from the start, but only a countable dense poset — e.g., spheres with rational centres and rational radii — one can take an ideal completion of the basis of intervals in the poset. The set of maximal elements of the completion, with Scott topology, is the "manifold", topologically; but there is no metric. (This is the fundamental problem of the causal set program.)

OK, and now what?

I don't know. There are other problems with this thought experiment. I'm sure you'll point out many.

Here's one: even if domain theory can be used to work around the underlying classical analysis, we may still end up with a "3D Brouwer-Weyl Continuum" that is too rigid in a technical sense.

Automorphisms of the causal order of the Minkowski space $\mathbb{R}^{1,n}$ for $n > 1$ are precisely the Lorenz transformations by a famous theorem of Alexandrov and Zeeman. In this sense, the structure of the continuum as a set of nested sequences of intervals (which would correspond to $n = 1$) seems to be fundamentally different from a higher-dimensional analog: the Alexandrov-Zeeman theorem does not hold for $n = 1$, as there are nonlinear bijections $\mathbb{R} \to \mathbb{R}$ that preserve interval order.

Thank you!