Aspects of Non-Associative, Non-Commutative Linear Logic

Eben Blaisdell, Max Kanovich, Stepan L. Kuznetsov,
Elaine Pimentel, and Andre Scedrov

Linear Logic and Its Variations

- Resource aware logics have been object of passionate study for quite some time now, with various motivations: usefulness for modelling computations, interesting algebraic semantics and nice proof-theoretic properties, polymodal extensions for specification of several behaviours, applications in linguistics, etc.
- Linear logic (LL), as introduced by Girard (1987), is a refinement of classical and intuitionistic logic. Structural rules of *contraction* and *weakening* in LL are allowed not for arbitrary formulae, but only for formulae under the *exponential*.
- Intuitionistic linear logic is denoted by ILL and uses only one exponential, !. In classical linear logic (LL), we have two dual exponentials, ! and ?.

Linear Logic and Its Variations

- LL and ILL still have two implicit structural rules which may be applied to all formulae, namely, *exchange* (commutativity) and *associativity*.
- The Lambek calculus LC, introduced by Lambek (1958) for linguistic applications, can be considered a non-commutative, but still associative version of ILL, but without the exponential and additive conjunction and disjunction.
  - The link between LC and ILL was noticed by Abrusci (1991).
  - In linear logic, one has to distinguish *multiplicative* conjunction and disjunction ($\otimes$, $⅋$) and *additive* ones ($\&$, $\oplus$).
- In this talk, we consider the *non-associative and non-commutative* intuitionistic linear logic, that is, the non-associative Lambek calculus.

Exponential

- The *exponential modality*, $!A$, of Girard's linear logic allows structural rules (weakening and contraction) for formulae under this modality.
- This modality allows embedding intuitionistic / classical logic into linear logic.
- However, the exponential leads to undecidability (Lincoln et al. 1992).
- It is important to notice that in the non-commutative case undecidability holds already for the multiplicative-only fragment, while in the commutative case this remains an open problem.

Subexponentials

- *Subexponentials* allow more fine-grained control over usage of structural rules.
- Namely, instead of one exponential ! we now have a family of subexponentials $!^i$, marked by subexponential labels $i \in I$.
  - I is a finite set of labels.
- Each subexponential allows a *subset* of the set of structural rules; this information is kept in the *subexponential signature* Σ.
- Another motivation for subexponentials is the *non-canonicity* of !: even with the same set of rules one can have several non-equivalent modalities.
- References for subexponentials: Danos, Joinet, Schellinx (1993), Nigam and Miller (2009); for the non-commutative case: Kanovich et al. (2019).

The Calculus acLL_{Σ}

- This talk is a sequel to the following talk on IJCAR 2022:

  E. Blaisdell, M. Kanovich, S. L. Kuznetsov, E. Pimentel, A. Scedrov. Non-associative, non-commutative multi-modal linear logic. In: Automated Reasoning, 11th International Conference, IJCAR 2022 (Haifa, Israel, August 8–10, 2022), LNCS vol. 13385, Springer, 2022, pp. 449–467.

- Our IJCAR 2022 talk features a subexponential extension of the non-associative Lambek calculus, denoted by acLL_{Σ}.
- This talk presents stronger undecidability results for fragments of this system.

The Calculus acLL_Σ

- The system, denoted by acLL_Σ, will be a sequent calculus with nested structures in antecedents.
- Possible structural rules will be contraction, weakening, associativity, and exchange, so we have the following set $\mathcal{A} = \{\mathsf{C}, \mathsf{W}, \mathsf{A1}, \mathsf{A2}, \mathsf{E}\}$.
  - In what follows, we shall have two rules for associativity, A1 and A2.
- The subexponential signature, or simply dependent multimodal logical system (SDML) is a triple $\Sigma = (I, \preccurlyeq, f)$, where $(I, \preccurlyeq)$ is a partially ordered set and $f : I \to 2^{\mathcal{A}}$.
- Upward closure: if $i \preccurlyeq j$, then $f(i) \subseteq f(j)$.

The Calculus acLL_Σ

- Elements of I will be labels for subexponentials.
- The subexponential $!^i$ allows structural rules from the set $f(i)$.
- The partial order $\preccurlyeq$ prescribes *interaction axioms* between subexponentials: if $i \preceq j$, then we have $!^j A \Rightarrow !^i A$.
- Actually, the form of interaction axioms is more general, as we shall see below.

The Calculus acLL_Σ

- Formulae of acLL_Σ are built from variables and constants 1 (unit) and $\top$ (additive truth), using the following binary operations: $\otimes$ (multiplicative conjunction), $\oplus$ (additive disjunction), $\&$ (additive conjunction), $\to$ (left implication), $\leftarrow$ (right implication), and unary subexponentials $!^i A$ ($i \in I$).
- Sequents of acLL_Σ are expressions of the form $\Gamma \Rightarrow A$, where A is a formula and Γ is a *nested structure.*
- Nested structures are defined by the following grammar: $\Gamma := F \mid (\Gamma, \Gamma) \mid \varnothing$.
- We shall consider *contexts,* which are nested structures with holes, of the form $\Gamma\{\ \}$ (one hole) or $\Gamma\{^1\ \}\ldots\{^n\ \}$ (several holes).
- Holes may be filled by formulae or other nested structures (in particular, the empty one).
- Structures with the empty structure are truncated in a natural way (e.g., $(\varnothing, \Gamma)$ is just Γ).

The Calculus acLL$_\Sigma$

Propositional rules

$$\frac{\Gamma\{(F,G)\} \Rightarrow H}{\Gamma\{F \otimes G\} \Rightarrow H}\ \otimes L \qquad \frac{\Gamma_1 \Rightarrow F \quad \Gamma_2 \Rightarrow G}{(\Gamma_1,\Gamma_2) \Rightarrow F \otimes G}\ \otimes R$$

$$\frac{\Gamma\{F\} \Rightarrow H \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{F \oplus G\} \Rightarrow H}\ \oplus L \qquad \frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \oplus F_2}\ \oplus R_i$$

$$\frac{\Gamma\{F_i\} \Rightarrow G}{\Gamma\{F_1 \,\&\, F_2\} \Rightarrow G}\ \& L_i \qquad \frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \,\&\, G}\ \& R$$

$$\frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(\Delta, F \to G)\} \Rightarrow H}\ \to L \qquad \frac{(F,\Gamma) \Rightarrow G}{\Gamma \Rightarrow F \to G}\ \to R$$

$$\frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(G \leftarrow F, \Delta)\} \Rightarrow H}\ \leftarrow L \qquad \frac{(\Gamma, F) \Rightarrow G}{\Gamma \Rightarrow G \leftarrow F}\ \leftarrow R$$

$$\frac{\Gamma\{\,\} \Rightarrow F}{\Gamma\{1\} \Rightarrow F}\ 1L \qquad \frac{}{\Rightarrow 1}\ 1R \qquad \frac{}{\Gamma \Rightarrow \top}\ \top R$$

The Calculus acLL_Σ

Initial and cut rules

$$\dfrac{}{F \Rightarrow F}\ \mathsf{init}$$

$$\dfrac{\Delta \Rightarrow F \quad \Gamma\left\{{}^{1}F\right\}\dots\{{}^{n}F\} \Rightarrow G}{\Gamma\left\{{}^{1}\Delta\right\}\dots\{{}^{n}\Delta\} \Rightarrow G}\ \mathsf{mcut}$$

The Calculus acLL_Σ

Subexponential rules

$$\frac{\Gamma^{\uparrow i} \Rightarrow F}{\Gamma \Rightarrow !^i F}\ !^i R$$

$$\frac{\Gamma\{F\} \Rightarrow G}{\Gamma\{!^i F\} \Rightarrow G}\ \mathrm{der}$$

Here $\Gamma^{\uparrow i}$ means that we require all formulae of Γ be either of the form $!^j A$, where $j \succcurlyeq i$ or of the form $!^k A$, where $f(k) \ni \mathrm{W}$ and $j \not\succcurlyeq i$, and remove the latter from Γ. (Otherwise $\Gamma^{\uparrow i}$ is undefined, and the rule is not applicable.)

The Calculus acLL_Σ

Structural rules

$$\frac{\Gamma\{(((!^a\Delta_1,\Delta_2),\Delta_3)\} \Rightarrow G}{\Gamma\{(!^a\Delta_1,(\Delta_2,\Delta_3))\} \Rightarrow G}\ \mathrm{A1} \qquad \frac{\Gamma\{(\Delta_1,(\Delta_2,!^a\Delta_3))\} \Rightarrow G}{\Gamma\{((\Delta_1,\Delta_2),!^a\Delta_3)\} \Rightarrow G}\ \mathrm{A2}$$

$$\frac{\Gamma\{(\Delta_2,!^e\Delta_1)\} \Rightarrow G}{\Gamma\{(!^e\Delta_1,\Delta_2)\} \Rightarrow G}\ \mathrm{E1} \qquad \frac{\Gamma\{(!^e\Delta_2,\Delta_1)\} \Rightarrow G}{\Gamma\{(\Delta_1,!^e\Delta_2)\} \Rightarrow G}\ \mathrm{E2}$$

$$\frac{\Gamma\{\,\} \Rightarrow G}{\Gamma\{!^w\Delta\} \Rightarrow G}\ \mathrm{W} \qquad \frac{\Gamma\left\{{}^1 !^c\Delta\right\}\ldots\left\{{}^n !^c\Delta\right\} \Rightarrow G}{\Gamma\left\{{}^1\ \right\}\ldots\left\{{}^k !^c\Delta\right\}\ldots\left\{{}^n\ \right\} \Rightarrow G}\ \mathrm{C}$$

Each rule is enabled only if the corresponding letter of $\mathcal{A} = \{\mathrm{A1}, \mathrm{A2}, \mathrm{E}, \mathrm{C}, \mathrm{W}\}$ belongs to $f(i)$, where $i = a, e, w, c$, respectively.

Linguistic Examples

- As already noticed, the Lambek calculus was introduced for modelling natural language syntax.
- The basic example is *"John likes Mary,"* which corresponds to the following sequent:

$$np, (np \to s) \leftarrow np, np \Rightarrow s.$$

  - Here *np* stands for "noun phrase" and *s* stands for "sentence." Further *n* will mean "common noun."
- This sequent keeps valid without associativity:

$$np, ((np \to s) \leftarrow np, np) \Rightarrow s.$$

Linguistic Examples

- Moreover, sometimes associativity leads to validating incorrect phrases.
- For example, phrases *"The Hulk is green"* and *"The Hulk is incredible"* [Moot, Retoré 2012] are validated by the following sequent:

$$(np \leftarrow n, n), ((np \rightarrow s) \leftarrow (n \leftarrow n), n \leftarrow n) \Rightarrow s.$$

- In the presence of associativity, the following is also derivable:

$$np \leftarrow n, n, (np \rightarrow s) \leftarrow (n \leftarrow n), n \leftarrow n, n \leftarrow n \Rightarrow s,$$

which corresponds to the incorrect phrase *"The Hulk is green incredible."*

Linguistic Examples

- Other syntactic phenomena, however, *require* associativity.
- An example is *"the girl whom John loves:"*

  $$np \leftarrow n, n, (n \rightarrow n) \leftarrow (s \leftarrow np), np, (np \rightarrow s) \leftarrow np \Rightarrow np.$$

  Here associativity is essential.
- In our subexponential extension of the non-associative Lambek calculus, we analyse this example by assigning to *whom* the formula $(n \rightarrow n) \leftarrow (s \leftarrow !^{a} np)$, where $f(a) = \{\mathrm{A2}\}$:

  $$np \leftarrow n, (n, ((n \rightarrow n) \leftarrow (s \leftarrow !^{a} np), (np, (np \rightarrow s) \leftarrow np))) \Rightarrow np.$$

Linguistic Examples

- The necessity of this more fine-grained control of associativity (instead of global associativity) is seen via a combination of these two examples.
- Phrases like *"The superhero whom Hawkeye killed was incredible"* and *"... was green"* are analysed using $!^a$:

$$(np \leftarrow n, (n, ((n \rightarrow n) \leftarrow (s \leftarrow !^a np), (np, (np \rightarrow s) \leftarrow np)))),$$
$$((np \rightarrow s) \leftarrow (n \leftarrow n), n \leftarrow n) \Rightarrow s.$$

- On the other hand, global non-associativity prevents from deriving incorrect phrases like *"The superhero whom Hawkeye killed was green incredible."*

Other Approaches

- There are also other approaches to controlling associativity and non-associativity in the Lambek calculus.
- The *multi-modal* Lambek calculus (Oehrle and Zhang 1989, Moortgat and Morrill 1991, Hepple 1994, Moot and Retoré 2012 etc) uses different *families of connectives*, distinguished by indices called *modes:* $\to_i, \leftarrow_i, \otimes_i, \ldots$ Each mode has its own set of rules.
- Another approach is the framework of the Lambek calculus with *brackets* (Morrill 1992, Moortgat 1994). This is a dual approach: the base system is associative, and brackets and bracket modalities introduce controlled non-associativity.

Undecidability Results

- In the commutative and associative case, LL (and ILL) is undecidable (Lincoln et al. 1992).
- However, this requires *additives;* (un)deciability of MELL, multiplicative-exponential linear logic, is a well-known open problem.
- Let us take a note that MELL with several (three is sufficient) subexponentials is undecidable (Chaudhuri 2014).
- In the non-commutative, but associative case, the exponential extension of the Lambek calculus is undecidable, even without additives.
- The latter undecidability result can be proved by encoding *reasoning from hypotheses* (non-logical axioms) in the Lambek calculus.
- The exponential may be replaced by an subexponential allowing non-local contraction (Kanovich et al. 2019).

Undecidability Results

- For the non-associative Lambek calculus, however, reasoning from hypotheses is polynomially (!) decidable (Buszkowski 2005).
- Thus we could make a conjecture that the multiplicative-only fragment of acLL_Σ, where Σ includes only one full-power exponential, is also decidable.
- However, additive operations or several subexponentials make the system undecidable.
- Thus, the situation is probably more like the *commutative associative* one.

Undecidability Results

Theorem

If there exists such $s \in I$ that $f(s) \ni \mathrm{C}$, then the derivability problem in acLL_Σ is undecidable. Moreover, this holds for the fragment with only $\otimes$, $\to$, $\oplus$, $!^s$.

- This result follows from undecidability of derivability from hypotheses (consequence relation) for the multiplicative-additive Lambek calculus (Chvalovský 2015). This result is a refinement of a result by Tanaka (2019).
- In IJCAR 2022: $f(s) \supseteq \{\mathrm{C}, \mathrm{W}\}$, i.e., weakening was also required.

Undecidability Results

Theorem

If there are $a, c \in I$ such that $f(a) = \{\mathrm{A1}, \mathrm{A2}\}$ and $f(c) \ni \mathrm{C}$, then the derivability problem in acLL_Σ is undecidable, in the fragment with only $\to$, $!^a$, $!^c$.

- This result is a purely multiplicative one. However, now we need two subexponentials, one for associativity, the other for contraction.
- We also use only one division, no product, using Buszkowski's (1982) ideas.

Going Classical

- We introduce the calculus CacLL_Σ, which is the **classical** (as Girard's original linear logic) extension of acLL_Σ.
- The motivation for going classical is in the line of De Groote and Lamarche (2002): we wish to observe the symmetries which are latent in the intuitionistic setting.
- Namely, intuitionistic linear logic systems lack multiplicative disjunction ($\invamp$) and negation.
- The classical system is symmetric... and in the substructural setting it is actually a **conservative extension** of the intuitionistic one!
  - ... up to a couple of exceptions.
  - A different (left-handed) one-sided classical system was proposed by Buszkowski (2016). Grammars based on Buszkowski's system generate context-free languages.

Formulae and One-Sided Sequents

- Sequents in the classical case are **one-sided**, of the form $\Rightarrow \Gamma$.
- Here Γ is a *structure*, that is, a binary tree of formulae:

$$\Gamma ::= \varnothing \mid F \mid (\Gamma, \Gamma).$$

- *Formulae* of CacLL_Σ are constructed as follows:

$$F, G, \ldots \; ::= \underbrace{A \mid A^\perp}_{\text{LITERALS}} \mid \underbrace{F \otimes G \mid 1 \mid F \parr G \mid \perp}_{\text{MULTIPLICATIVES}} \mid \underbrace{F \oplus G \mid 0 \mid F \,\&\, G \mid \top}_{\text{ADDITIVES}} \mid \underbrace{!^i F \mid ?^i F}_{\text{SUBEXP.}}$$

- *Subexponential signature:* $\Sigma = (\mathcal{I}, \preceq, f)$, where $f(i) \subseteq \{\mathsf{C}, \mathsf{W}, \mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$ for each $i \in \mathcal{I}$.
  - The f function declares which structural rules are available for a given subexponential; $\preceq$ defines entailments between subexponentials.

Rules of CacLL_Σ

Propositional rules

$$\frac{\Rightarrow \Gamma, G \quad \Rightarrow \Delta, F}{\Rightarrow ((\Gamma, \Delta), F \otimes G)} \otimes \qquad \frac{\Rightarrow \Gamma\{(F, G)\}}{\Rightarrow \Gamma\{F \parr G\}} \parr$$

$$\frac{\Rightarrow \Gamma\{F_i\}}{\Rightarrow \Gamma\{F_1 \oplus F_2\}} \oplus_i \qquad \frac{\Rightarrow \Gamma\{F_1\} \quad \Rightarrow \Gamma\{F_2\}}{\Rightarrow \Gamma\{F_1 \,\&\, F_2\}} \&$$

$$\frac{\Rightarrow \Gamma\{\,\}}{\Rightarrow \Gamma\{\bot\}} \bot \qquad \frac{}{\Rightarrow 1} 1 \qquad \frac{}{\Rightarrow \Gamma\{\top\}} \top$$

Structural Rules

$$\frac{\Rightarrow (\Delta, \Gamma)}{\Rightarrow (\Gamma, \Delta)} \mathrm{E} \qquad \frac{\Rightarrow (\Gamma, (\Delta, \Pi))}{\Rightarrow ((\Gamma, \Delta), \Pi)} \mathrm{A1} \qquad \frac{\Rightarrow ((\Gamma, \Delta), \Pi)}{\Rightarrow (\Gamma, (\Delta, \Pi))} \mathrm{A2}$$

Initial and cut rules

$$\frac{}{\Rightarrow (A, A^\perp)} \mathrm{init} \qquad \frac{\Rightarrow (\Gamma, A) \quad \Rightarrow (A^\perp, \Delta)}{\Rightarrow (\Gamma, \Delta)} \mathrm{cut}$$

Rules of CacLL_Σ

Subexponential rules

$$\frac{\Rightarrow (\Gamma^{\uparrow i}, F)}{\Rightarrow (\Gamma, !^i F)}\ \mathrm{prom} \qquad \frac{\Rightarrow \Gamma\{F\}}{\Rightarrow \Gamma\{?^i F\}}\ \mathrm{der}$$

Subexponential Structural rules

$$\frac{\Rightarrow (((\Delta_1, \Delta_2), \Delta_3), ?^{a1}\Gamma)}{\Rightarrow ((\Delta_1, (\Delta_2, \Delta_3)), ?^{a1}\Gamma)}\ ?\mathrm{A1} \qquad \frac{\Rightarrow ((\Delta_1, (\Delta_2, \Delta_3)), ?^{a2}\Gamma)}{\Rightarrow (((\Delta_1, \Delta_2), \Delta_3), ?^{a2}\Gamma)}\ ?\mathrm{A2}$$

$$\frac{\Rightarrow ((\Delta_2, \Delta_1), ?^e\Gamma)}{\Rightarrow ((\Delta_1, \Delta_2), ?^e\Gamma)}\ ?\mathrm{E}$$

$$\frac{\Rightarrow \Gamma\{\,\}}{\Rightarrow \Gamma\{?^w\Delta\}}\ ?\mathrm{W} \qquad \frac{\Rightarrow \Gamma\{?^c\Delta\}\ldots\{?^c\Delta\}}{\Rightarrow \Gamma\{\,\}\ldots\{?^c\Delta\}\ldots\{\,\}}\ ?\mathrm{C}$$

Here $\Gamma^{\uparrow i}$ means the following: Γ may be weakened, and after that we guarantee that it consists of $!^j G$'s, where $j \succeq i$.

Cyclic Shifts and Keyrings

- Global structural rules, E, A1 and A2, may look strange, as we did not want our system to be commutative or associative.
- However, they are in fact not "real" commutativity and associativity, but rather a non-associative version of cyclic reorganisation of sequent structure.
- In the associative, but non-commutative setting, there is *cyclic linear logic,* which allows a global rule of cyclic shift (E).
- This rule might be made implicit, by considering cyclically ordered sequences of formulae as sequent structures.

Cyclic Shifts and Keyrings

- For our non-associative system, such an invariant description of structures is more sophisticated.
- This construction is called *unrooted cyclically-ordered-neigbor 3-regular trees with leaves:* each internal node keeps the cyclic order of its neighbours, but nothing more.

Cut Elimination

Theorem

If a sequent $\Rightarrow \Gamma$ *is provable in* CacLL_{Σ}*, then there is a proof in which the* cut *rule is not applied.*

Cut Elimination

Theorem

If a sequent $\Rightarrow \Gamma$ *is provable in* CacLL_{Σ}, *then there is a proof in which the* cut *rule is not applied.*

The proof uses the classical Gentzen method. Contraction is handled using the mix rule:

$$\frac{\Rightarrow (\Gamma, !^c A^{\perp}) \quad \Rightarrow (?^c A, \Delta\{?^c A\} \ldots \{?^c A\})}{\Rightarrow (\Gamma, \Delta\{\} \ldots \{\})}\ \mathsf{mix}$$

Embedding of acLL_Σ into CacLL_Σ

As noticed before, the intuitionistic system is conservatively embedded into the classical one, via the following translation:

$$\widehat{p} :\equiv p \qquad \widehat{A \otimes B} :\equiv \widehat{A} \otimes \widehat{B}$$

$$\widehat{A \to B} :\equiv \widehat{A}^{\perp} \parr \widehat{B} \qquad \widehat{B \leftarrow A} :\equiv \widehat{B} \parr \widehat{A}^{\perp}$$

$$\widehat{A \,\&\, B} :\equiv \widehat{A} \,\&\, \widehat{B} \qquad \widehat{A \oplus B} :\equiv \widehat{A} \oplus \widehat{B}$$

$$\widehat{!^i A} :\equiv !^i \widehat{A} \qquad \widehat{1} :\equiv 1$$

$$\widehat{\top} :\equiv \top \qquad \widehat{\Gamma \Rightarrow A} :\equiv (\widehat{\Gamma}^{\perp}, \widehat{A})$$

(Negations in our calculus are tight, so applying negation to a formula or structure actually means propagation by de Morgan laws, exchanging the order for multiplicatives.)

Conservativity

Theorem

If for all labels i in Σ we have $f(i) \subseteq \{\mathsf{C}, \mathsf{W}, \mathsf{E}\}$, then a sequent $\Gamma \Rightarrow A$ is provable in acLL_Σ *iff* $\Rightarrow (\widehat{\Gamma}^{\perp}, \widehat{A})$ *is provable in* CacLL_Σ*.*

- This theorem is in the spirit of Schellinx' result (1991) on embedding intuitionistic LL into classical LL.
- The proof uses a technique by Pentus (1998).
- For modalities licensing associativity, the counter-example is $((a \otimes b) \otimes !^a c) \Rightarrow (a \otimes (b \otimes !^a c))$.
- This counterexample comes from the difference in associativity rules for acLL_Σ and CacLL_Σ. It can be eliminated by extending acLL_Σ with appropriate rules.

Counterexample with Zero

- Adding the zero constant to acLL_Σ also ruins conservativity:

Theorem
The following sequent is not provable in acLL_Σ, *but its translation is provable in* CacLL_Σ*:*

$$!^a((r \leftarrow (0 \rightarrow q)) \leftarrow p), (s \leftarrow p) \rightarrow 0 \Rightarrow r.$$

- This is a modification of the counterexample by Schellinx.

Future Work

- Focusing for CacLL_Σ.
- Connections to other frameworks with controlled associativity.
- Complexity: undecidability for the whole CacLL_Σ gets inherited from acLL_Σ by conservativity, but decidability results for fragments would be stronger.

Hvala!