On properties of feasibility in nonstandard Heyting arithmetic

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Assume we have an arithmetical theory with

- constant symbol 0,
- function symbols S, $+$, $\cdot$ (and possibly other symbols for primitive recursive functions)
- predicate symbols $<$, $=$.

In the theory, there are

axioms for equality,

axioms for the predicate symbol $<$, namely,

- $\forall x \neg (x < x)$, (irreflexivity)
- $\forall xyz(x < y \wedge y < z \supset x < z)$, (transitivity)
- $\forall xy(x < y \vee x = y \vee y < x)$ (trichotomy)
- $\forall xyz(x < y \supset x + z < y + z)$, ($+$ respects the order)
- $\forall xyz(0 < z \wedge x < y \supset x \cdot z < y \cdot z)$, ($\cdot$ respects the order)
- $\forall xy \exists z(x < y \supset x + z = y)$ (subtraction)
- $0 < 1 \wedge \forall x(0 < x \supset x = 1 \vee 1 < x)$, (discrete structure axiom)
- $\forall x(x = 0 \vee 0 < x)$. (0 is the smallest element)

Observe that the usual axioms for the successor function are derivable from the above ones

- $\forall x \neg S(x) = 0$,
- $\forall x \forall y (S(x) = S(y) \supset x = y)$.

In addition, we have defining axioms for all primitive recursive functions occurring in the language and axiom for induction, which is to be discussed later. We write $x + 1$ for $S(x)$.

Heyting arithmetic

We choose a new constant c and add the axioms expressing that c is an infinite element:

$$\overline{n} < c,$$

where $\overline{n} = S \dots S(0)$ with the number of S's being equal to n. When we augment the theory with induction

$$A(0) \wedge \forall x(A(x) \supset A(x+1)) \supset \forall x A(x) \quad (Ind)$$

we obtain HA^c.

The underlying logic is first-order intuitionistic logic. This means that we replace the axiom $\neg\neg A \supset A$ with $\bot \supset A$. Some statements we are used to in classical logic will not be provable in this case. For example:

- $A \vee \neg A$
- $\neg(A \wedge B) \supset \neg A \vee \neg B$
- $(A \supset B) \supset \neg A \vee B$
- $(\forall x A(x) \supset B) \supset \exists x(A(x) \supset B)$, if $x \notin fv(B)$
- $\neg\forall x A(x) \supset \exists x \neg A(x)$
- $\neg\neg\exists x A(x) \supset \exists x A(x)$
- $(\neg A \supset \exists x B(x)) \supset \exists x(\neg A \supset B(x))$, where $fv(A) = \emptyset$ and $fv(B) = \{x\}$

At the same time, let

- $P^g \rightleftharpoons \neg\neg P$ if P is atomic,
- $(A \wedge B)^g \rightleftharpoons A^g \wedge B^g$,
- $(A \supset B)^g \rightleftharpoons A^g \supset B^g$,
- $(A \vee B)^g \rightleftharpoons \neg(\neg A^g \wedge \neg B^g)$,
- $(\forall x A)^g \rightleftharpoons \forall x A^g$,
- $(\exists x A)^g \rightleftharpoons \neg\exists x \neg A^g$.

Theorem 1

Let A be a negative formula, that is, the atomic formulae of A are negated. Then

$$\vdash_m A \equiv \neg\neg A.$$

Theorem 2

Let A be a formula. Then

1. $\vdash_c A \equiv A^g$,
2. $\Gamma \vdash_c A \Leftrightarrow \Gamma^g \vdash_m A^g$, *where* $\vdash_c$ *and* $\vdash_m$ *mean provability in classical and in minimal logic, respectively.*

A^g *is called the Gödel-Gentzen translation of* A.

Theorem 3

Let A be negative. Then

$$\vdash_c A \Leftrightarrow \vdash_i A^g.$$

Interestingly, HA^c is conservative over HA and this can be justified easily.

Theorem 4 (Martin-Löf)

Let A be a formula of HA. Assume $HA^c \vdash A$. Then there is a natural number n such that $HA \vdash (\forall x \geq n)(A[c/x])$.

Theorem 5 (Transfer principle)

Let A be a formula of HA. Then $HA \vdash A \Leftrightarrow HA^c \vdash A$.

However, not all properties of HA are preserved by HA^c.

Definition

Let T be a first-order theory.

1. We say that T enjoys the disjunction property if $T \vdash B \vee C$ implies $T \vdash B$ or $T \vdash C$ for any closed formula $B \vee C$.
2. Similarly, T has the existence property if, for any closed formula $\exists x A(x)$, $T \vdash \exists x A(x)$ implies $T \vdash A(t)$ for some term t.

Fact

HA has both the disjunction and the existence properties.

This is not true in HA^c.

Proposition 1 (Martin-Löf)

There are closed atomic formulae A and B of HA^c such that $HA^c \vdash A \vee B$, but $HA^c \nvdash A$ and $HA^c \nvdash B$.

An analogous statement holds for the existential property.

Next, we extend our theory with a one-place predicate symbol F expressing feasibility:

(F1) $F(0)$,

(F2) $\forall x \forall y (F(x) \wedge y < x \supset F(y))$,

(F3) $\forall x (F(x) \supset x < c)$,

(F4) $\forall x_1 \ldots \forall x_n (F(x_1) \wedge \ldots \wedge F(x_n) \supset F(g(x_1, \ldots, x_n)))$, for each function symbol g.

For the induction, we introduce

$$A(0) \wedge \forall^f x(A(x) \supset A(Sx)) \supset \forall^f x A(x) \quad (Ind^f),$$

where $A(x)$ does not contain F and the notation $\forall^f x A(x)$ stands for $\forall x(F(x) \supset A(x))$. We denote the theory obtained in this way by HAF.

We provide a simple argument for the conservativity of HAF.

Theorem 6

HAF is conservative over HA. Namely, let A be a closed formula of HA. Then $HAF \vdash A$ implies $HA \vdash A$.

Proof

Assume $HAF \vdash A$ and $HA \nvdash A$. By the conservativity of HA^c, we have that $HA^c \nvdash A$. This means that $HA^c + \neg A$ is consistent hence, there exists a model M for $HA^c + \neg A$. We define an extension of M: we put $MF \models F(a)$ for an object a of the model M if there exists a natural number m such that $M \models (a = \overline{m})$. On the other hand, $MF \models HAF$ and $MF \models \neg A$, a contradiction.

In every model of HAF, the feasible elements form a proper cut, namely, a set downward closed and, for every member, it contains the successor element also. Let us restrict our attention to HA^c for the moment, and let $M \models HA^c$. Then $I \subseteq M$ is a cut (or initial segment) if, for any $x, y \in M$, $x \in I$ and $y < x$ imply $y \in I$, and $x \in I$ involves $x + 1 \in I$. A cut is proper if $I \neq M$. By a theorem of A. Robinson, in nonstandard Peano arithmetic, proper cuts cannot be defined by a formula. In this respect, F is different, since it is not a formula of HA. We provide first a weaker version of HA in which we verify that feasible elements are exactly the numerals.

Heyting arithmetic with feasibility

Let us make the following changes to the predicate calculus. We exchange

$$\forall x A(x) \supset A(t) \quad \text{with} \quad \forall x A(x) \wedge F(t) \supset A(t), \text{ and}$$

$$A(t) \supset \exists x A(x) \quad \text{with} \quad A(t) \wedge F(t) \supset \exists x A(x).$$

In effect, every term introduced in a derivation is a feasible element by this stipulation. The new theorem will be denoted by HAF^*.

Theorem 7

Assume $HAF^* \vdash F(t)$ *for a closed term* t*. Then there exists an* n *such that* $HAF^* \vdash t = \overline{n}$.

We introduce the following notation, stemming from S. C. Kleene:

- $T_n(e, x_1, ..., x_n, z)$: e is the code of a partial recrsive function of arguments $x_1, \ldots, x_n$, and z is the code for the specific computation,
- $U(z)$: the result of the computation coded by z. U itself is a primitve recursive function.

Furthermore,

- $\{e\}(m_1, ..., m_n) = k \rightleftharpoons \exists z(T_n(e, m_1, ..., m_n, z) \wedge U(z) = k)$,
- $!\{e\}(m_1, ..., m_n) \rightleftharpoons \exists y\{e\}(m_1, ..., m_n = y)$,
- $!\{n\}(m) \wedge \mathcal{A}(\{n\}(m)) \rightleftharpoons \exists v(\{n\}(m) = v) \wedge \forall v(\{n\}(m) = v \supset \mathcal{A}(v))$, where $\mathcal{A}$ is a relation over natural numbers.

In the sequel, we will be interested only in closed formulas of HAF^*. Accordingly, the axioms are the universal closures of the axioms discussed so far and the only inference rule is $\forall(A \supset B)$ and $\forall A$ implies $\forall B$, where $\forall A$ denotes the universal closure of formula A.

Let $\vdash$ denote provability in HAF^*. Let us define

- $nrt = s \rightleftharpoons Prf(n, \ulcorner t = s \urcorner)$, where $n \in \mathbb{N}$, and Prf is a primitive recursive relation coding provability in HAF^*,
- $nrt < s \rightleftharpoons Prf(n, \ulcorner t < s \urcorner)$,
- $nrF(t) \rightleftharpoons j_1(n)rt = \overline{j_2(n)}$, where t is a closed term.

HAF^*

- $nrA \wedge B \rightleftharpoons j_1(n)rA \wedge j_2(n)rB$,
- $nrA \vee B \rightleftharpoons (j_1(n) = 0 \supset j_1(j_2)(n)rA) \wedge (j_1(n) \neq 0 \supset j_2(j_2)(n)rB)$,
- $nrA \supset B \rightleftharpoons \forall m(mrA \supset !\{n\}(m) \wedge \{n\}(m)rB)$ and $\vdash A \supset B$,
- $nr\forall xA(x) \rightleftharpoons \forall m(!\{n\}(m) \wedge \{n\}(m)rA(\overline{m}))$ and $\vdash \forall xA(x)$,
- $nr\exists xA(x) \rightleftharpoons j_1(n)rA(\overline{j_2(n)})$.

Lemma 1

Let $A(t)$ be a closed formula, and s be a closed term. Assume $\vdash t = s$ and $\exists n(nrA(t))$. Then $\exists m(mrA(s))$.

Theorem 8

Let A be a closed formula. Then $\vdash A$ if and only if $\exists n(nrA)$.

Proof

($\Rightarrow$): We have to check that the assertion is true for the axioms of the predicate calculus and for the inference rule, as well. Since the axioms are the universal closure of the usual axioms, assuming $\forall A$ is an axiom, one needs to verify the realizability of every possible substitution $A(x_1/\overline{n_1}, \ldots, x_k/\overline{n_k})$, where $fv(A) = \{x_1, \ldots, x_k\}$ and $\{\overline{n_1}, \ldots, \overline{n_k}\}$ are numerals. We omit such details and assume that the instances of the axioms we are checking formulas that do not commence with universal quantifiers.

Proof

Regarding the inference rule $\forall(A \supset B)$ and $\forall A$ implies $\forall B$, suppose $nr\forall(A \supset B)$ and $kr\forall A$. Then $!\{n\}(\overline{m_1}, \ldots, \overline{m_p}) \wedge \{n\}(\overline{m_1}, \ldots, \overline{m_p})r(A \supset B)'$ and $!\{k\}(\overline{m'_1}, \ldots, \overline{m'_q}) \wedge \{k\}(\overline{m'_1}, \ldots, \overline{m'_q})rA''$, where $(A \supset B)'$ and A'' are obtained from $A \supset B$ and A by substituting $\overline{m_1}, \ldots, \overline{m_p}$ and $\overline{m'_1}, \ldots, \overline{m'_q}$, respectively, in place of the parameters. Let $\{\overline{m'_1}, \ldots, \overline{m'_q}\} \subseteq \{\overline{m_1}, \ldots, \overline{m_p}\}$ and let $\{\overline{m''_1}, \ldots, \overline{m''_p}\}$ be the numerals substituted for the variables in $fv(B)$. Let d be the primitive recursive function such that $!\{d\}(\overline{m''_1}, \ldots, \overline{m''_s})$ if and only if $!(\{n\}(\overline{m_1}, \ldots, \overline{m_p}))(\{k\}(\overline{m'_1}, \ldots, \overline{m'_q}))$ and, in this case, the two values are equal. Then $\{d\}(\overline{m''_1}, \ldots, \overline{m''_s})rB'$.

Proof

The assertion for the axioms of the predicate calculus can be verified following the pattern of the original proof of Kleene[1]. For example:

- $(A \supset C) \supset (B \supset C) \supset (A \vee B \supset C)$.
  Let $nr(A \supset C)$, $kr(B \supset C)$ s $lr(A \vee B)$. We have to distinguish two cases:
  - $j_1(l) = 0$: $j_1(j_2(l))rA$, whence $\{n\}(j_1(j_2(l)))rC$,
  - $j_1(l) \neq 0$: $j_2(j_2(l))rB$, whence $\{k\}(j_1(j_2(l)))rC$.

[1] S. C. Kleene: *Introduction to Metamathematics*, North Holland, Amsterdam, 1952.

Proof

- $\forall x A(x) \wedge F(t) \supset A(t)$.
  Assume $nr\forall x A(x)$ and $krF(t)$. Then $j_1(k)rt = \overline{j_2(k)}$. Since $n(\overline{j_2(k)})rA(\overline{j_2(k)})$, the result follows by applying Lemma ??.
- Induction axiom.
  Suppose $nrA(0) \wedge \forall^f x(A(x) \supset A(Sx))$. We would like to find u such that $!\{u\}(n) \wedge \{u\}(n)r\forall^f xA(x)$. Let k be arbitrary and v be such that $vrF(\overline{k})$. A p must be found such that, for each v like that, $!\{p\}(v)$ and $\{p\}(v)rA(\overline{k})$. Since k is a natural number, we can argue by induction on k.

Proof

By assumption, $j_1(n) r A(\mathit{0})$. Let $s_0 = j_1(n)$, and assume $m_0 \, r F(\mathit{0})$. Then

$$\underbrace{\{\{\{j_2(n)\}(0)\}(m_0)\}(j_1(n))}_{s_1} \, rA(\overline{1}).$$

Next, let $m_1 \, rF(\overline{1})$. Then

$$\underbrace{\{\{\{j_2(n)\}(1)\}(m_1)\}(s_1)}_{s_2} \, rA(\overline{2}).$$

Continuing in this way, we can find an s_k such that $s_k \, rA(\overline{k})$.

Proof

- $\forall x(F(x) \supset x < c)$.
  Suppose $n r F(\overline{m})$ for some n and m. Then $\overline{Prf(j_1(n), \ulcorner \overline{m} = j_2(n) \urcorner)}$. This means that $\overline{m}$ is a numeral, from which $\overline{m} < c$. Hence, $\exists k(k r \overline{m} < c)$.

($\Leftarrow$): By induction on A.

The proof of Theorem ??

Let us assume $HAF^* \vdash F(t)$. Then Theorem **??** provides us with an n such that $nrF(t)$. This means that $Prf(j_1(n), \ulcorner \overline{j_2(n)} = t \urcorner)$, from which the statement follows.

DP and EP in HAF^*

Corollary 1

Let A be a closed formula. Assume $\vdash A$. Then the following statements are valid:

1. *If $A = (B \vee C)$, then $\vdash B$ or $\vdash C$.*
2. *If $A = \exists x C(x)$, then there exists k such that $\vdash C(\overline{k})$.*

Proof

We verify the second statement. Assume $\vdash \exists x C(x)$. By the previous theorem, $nr\exists x C(x)$ for some n. Then $j_1(n) r C(\overline{j_2(n)})$, from which $\vdash C(\overline{j_2(n)})$ follows.

Let us examine some of the implications Corollary **??** induces in *HAF*. Firstly, we provide some connections between derivations in *HAF* and HAF^*.

Proposition 2

Let A be a closed formula such that $HAF \vdash A$. Then there are closed terms $t_1, \ldots, t_k$ in A such that $HAF^* \vdash F(t_1) \wedge \ldots \wedge F(t_k) \supset A$.

Intuitively, $t_1, \ldots, t_k$ are the terms introduced by the axiom $\forall x A(x) \supset A(t)$ in a *HAF*-derivation. The reverse statement is straightforward.

Proposition 3

Let $HAF^* \vdash A$ for some closed formula A. Then $HAF \vdash A$.

As a consequence, we assert a form of disjunction and existential properties for *HAF*.

Corollary 9

Let A be a provable formula of HAF such that c does not occur in A. Then the following statements hold true:

1. *If* $A = B \vee C$, *then* $HAF \vdash B$ *or* $HAF \vdash C$.
2. *If* $A = \exists x C(x)$, *then* $HAF \vdash C(\overline{k})$ *for some natural number k.*

Proof

We examine only the second item. Assume $HAF \vdash \exists x C(x)$. Then, by Proposition **??**, $HAF^* \vdash \bigwedge_{i=1}^{k} F(t_i) \supset \exists x C(x)$ for some closed terms $\{t_1, \ldots, t_k\}$. By assumption, $\vdash \bigwedge_{i=1}^{k} F(t_i)$, which implies $HAF^* \vdash \exists x C(x)$. Corollary **??** provides a numeral $\overline{k}$ such that $HAF^* \vdash C(\overline{k})$. Proposition **??** yields the result.

Open problems

- Let us formalize Church's thesis in the following manner:

$$(CT^f)\quad \forall x \exists^f y A(x,y) \supset \exists^f e \forall x \exists^f y (T(e,x,y) \wedge A(x,Uy)).$$

  Is CT^f consistent with HAF?

- If we formalize Church's thesis in the usual way, namely,

$$(CT)\quad \forall x \exists y A(x,y) \supset \exists e \forall x \exists y (T(e,x,y) \wedge A(x,Uy))$$

  such that CT is now a formula schema in HAF, what can be said about the consistency of $HAF + CT$?

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