Interpolation Properties of Proofs with Cuts

Anela Lolić
TU Wien

LAP 2024, Dubrovnik

Maehara's partition method

Definition (LK#)

Gentzen's system LK + the predicate symbol $\top$ with 0 argument places and the additional axiom $\vdash \top$.

Definition (partition)

Let $S: \; \Gamma \vdash \Delta$ be a sequent and let Γ_1, Γ_2 be a permutation variant of Γ, and Δ_1, Δ_2 a permutation variant of Δ. Then $[\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ is called a partition of S.

Maehara's partition method

Definition (LK#)

Gentzen's system LK + the predicate symbol $\top$ with 0 argument places and the additional axiom $\vdash \top$.

Definition (partition)

Let $S: \ \Gamma \vdash \Delta$ be a sequent and let Γ_1, Γ_2 be a permutation variant of Γ, and Δ_1, Δ_2 a permutation variant of Δ. Then $[\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ is called a partition of S.

Example

Consider the sequent $\forall x(P(x) \to Q(x)) \vdash \exists y(P(a) \to Q(y))$.

$$X: [\{\forall x(P(x) \to Q(x)); \ \}, \{ \ ; \exists y(P(a) \to Q(y))\}]$$

Weak interpolant

Definition

Let S be a sequent and $X : [\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ a partition of S. Then the formula I is called a weak interpolant of S w.r.t. X if

1. φ_1 is an LK#-proof of $\Gamma_1 \vdash \Delta_1, I$, and
   φ_2 is an LK#-proof of $I, \Gamma_2 \vdash \Delta_2$.
2. The predicate symbols in I are a subset of the predicate symbols occurring in $\{\Gamma_1, \Delta_1\}$ and $\{\Gamma_2, \Delta_2\}$.

A proof of the form

$$\frac{\begin{array}{c}(\varphi_1)\\ \Gamma_1 \vdash \Delta_1, I\end{array} \qquad \begin{array}{c}(\varphi_2)\\ I, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

is called an interpolation derivation for S w.r.t. X.

Strong interpolant

weak interpolant + all its free variables, constant symbols and function symbols occur in $\{\Gamma_1, \Delta_1\}$ and $\{\Gamma_2, \Delta_2\}$.

Strong interpolant

weak interpolant + all its free variables, constant symbols and function symbols occur in $\{\Gamma_1, \Delta_1\}$ and $\{\Gamma_2, \Delta_2\}$.

Lemma

Let I be a weak interpolant of a sequent S w.r.t. a partition X. Then there exists an strong interpolant I^* of S w.r.t. X.

Proof.

$X : [\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ partition of S, and the corresponding weak interpolation derivation $\psi =$

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma_1 \vdash \Delta_1, I\end{array} \qquad \begin{array}{c}(\varphi_2)\\ I, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

Select a maximal term t not occurring in both partitions, assume $t = f(t_1, \ldots, t_n)$ for some function symbol f not occurring as a function symbol in both partitions. We distinguish

Proof.

$X : [\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ partition of S, and the corresponding weak interpolation derivation $\psi =$

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma_1 \vdash \Delta_1, I\end{array} \qquad \begin{array}{c}(\varphi_2)\\ I, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

Select a maximal term t not occurring in both partitions, assume $t = f(t_1, \ldots, t_n)$ for some function symbol f not occurring as a function symbol in both partitions. We distinguish

1. f occurs only in $\{\Gamma_1; \Delta_1\}$.

$$\dfrac{\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma_1 \vdash \Delta_1, I\end{array}}{\Gamma_1 \vdash \Delta_1, \exists x I\{t \leftarrow x\}}\ \exists_r \qquad \dfrac{\begin{array}{c}(\varphi_2\{t \leftarrow \alpha\})\\ I\{t \leftarrow \alpha\}, \Gamma_2 \vdash \Delta_2\end{array}}{\exists x I\{t \leftarrow x\}, \Gamma_2 \vdash \Delta_2}\ \exists_l}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

Proof.

$X : [\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ partition of S, and the corresponding weak interpolation derivation $\psi =$

$$\dfrac{\begin{matrix}(\varphi_1) \\ \Gamma_1 \vdash \Delta_1, I\end{matrix} \qquad \begin{matrix}(\varphi_2) \\ I, \Gamma_2 \vdash \Delta_2\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

Select a maximal term t not occurring in both partitions, assume $t = f(t_1, \ldots, t_n)$ for some function symbol f not occurring as a function symbol in both partitions. We distinguish

2. f occurs only in $\{\Gamma_2; \Delta_2\}$.

$$\dfrac{\dfrac{\begin{matrix}(\varphi_1\{t \leftarrow \alpha\}) \\ \Gamma_1 \vdash \Delta_1, I\{t \leftarrow \alpha\}\end{matrix}}{\Gamma_1 \vdash \Delta_1, \forall x I\{t \leftarrow x\}}\ \forall_r \qquad \dfrac{\begin{matrix}(\varphi_2) \\ I, \Gamma_2 \vdash \Delta_2\end{matrix}}{\forall x I\{t \leftarrow x\}, \Gamma_2 \vdash \Delta_2}\ \forall_l}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

Proof.

$X : [\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ partition of S, and the corresponding weak interpolation derivation $\psi =$

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma_1 \vdash \Delta_1, I\end{array} \qquad \begin{array}{c}(\varphi_2)\\ I, \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

Select a maximal term t not occurring in both partitions, assume $t = f(t_1, \ldots, t_n)$ for some function symbol f not occurring as a function symbol in both partitions. We distinguish

3. f does not occur in the partitions. Then the interpolant can be constructed as in case 1 or in case 2, as both constructions work.

Lemma (Maehara's lemma)

Let $\Gamma \vdash \Delta$ *be* **LK***-provable, and* $X : [\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}]$ *an arbitrary partition of* $\Gamma \vdash \Delta$*. Then there exists a formula* I*, called the interpolant of* $\Gamma \vdash \Delta$ *w.r.t. the partition* X*, s.t.*

1. $\Gamma_1 \vdash \Delta_1, I$ *and* $I, \Gamma_2 \vdash \Delta_2$ *are both* LK#*-provable.*
2. I *contains only free variables and individual and predicate constants apart from* $\top$ *that occur in* $\{\Gamma_1; \Delta_1\}$ *and* $\{\Gamma_2; \Delta_2\}$*.*

Proof.

By induction on the number of inferences k in a cut-free proof of $\Gamma \vdash \Delta$.

- Interpolants I s.t. $\Gamma \vdash I$ and $I \vdash \Delta$ are provable.
- $k = 0$. The sequent $\Gamma \vdash \Delta$ is an axiom of the form $C \vdash C$. We look at the partition $[\{C;\}, \{;C\}]$. C fulfills all requirements for an interpolant of $C \vdash C$.
- $k > 0$: Consider the last inference in the derivation.

Proof.

Let $\forall_l$ be the last inference:

$$\frac{F(t), \Gamma \vdash \Delta}{\forall x F(x), \Gamma \vdash \Delta}\ \forall_l$$

with partition $[\{\forall x F(x), \Gamma; \}, \{; \Delta\}]$.
The partition in the upper sequent is $[\{F(t), \Gamma; \}, \{; \Delta\}]$.
By IH:

$$F(t), \Gamma \vdash I(b_1, \ldots, b_n) \text{ and } I(b_1, \ldots, b_n) \vdash \Delta.$$

$b_1, \ldots, b_n$ the free variables and constants occurring in t.
Replace the $b_{i_1}, \ldots, b_{i_m}$ that do not occur in $F(x), \Delta$ by bound variables $y_1, \ldots, y_m$:

$$I' = \forall y_1, \ldots, \forall y_m I(b_1, \ldots, y_1, \ldots, y_m, \ldots, b_n)$$

$$\cfrac{\cfrac{\cfrac{F(t), \Gamma \vdash I(b_1, \ldots, b_n)}{\forall x F(x), \Gamma \vdash I(b_1, \ldots, b_n)}\ \forall_l}{\forall x F(x), \Gamma \vdash I'}\ \forall_r \qquad \cfrac{I(b_1, \ldots, b_n) \vdash \Delta}{I' \vdash \Delta}\ \forall_l}{\forall x F(x), \Gamma \vdash \Delta}\ cut$$

Proof.

Let $\forall_r$ be the last inference:

$$\frac{\Gamma \vdash \Delta, F(y)}{\Gamma \vdash \Delta, \forall x F(x)} \; \forall_r$$

with partition $[\{\Gamma;\}, \{;\Delta, \forall x F(x)\}]$.
The partition in the upper sequent is $[\{\Gamma;\}, \{;\Delta, F(y)\}]$.
By IH:

$$\Gamma \vdash I \text{ and } I \vdash \Delta, F(y).$$

Since y does not occur in $\Gamma, \Delta, F(x)$ it does not occur in I and we infer

$$\frac{I \vdash \Delta, F(y)}{I \vdash \Delta, \forall x F(x)} \; \forall_r$$

I is also an interpolant for the lower sequent $\Gamma \vdash \Delta, \forall x F(x)$.

Craig's interpolation theorem

Theorem

Let A and B be formulas s.t. $A \to B$ is **LK***-provable.*
If A and B have at least one predicate constant in common, then there is a formula I, called the interpolant of A and B, s.t.

- *I only contains free variables and individual and predicate constants that occur in both A and B,*
- *and $A \to I$ and $I \to B$ are* **LK***-provable.*

If A and B have no predicate constant in common, then either $A \to$ or $\to B$ is **LK***-provable.*

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \qquad Q(a) \vdash Q(a)}{P(a), P(a) \to Q(a) \vdash Q(a)} \to_l}{P(a) \to Q(a) \vdash P(a) \to Q(a)} \to_r}{P(a) \to Q(a) \vdash \exists y(P(a) \to Q(y))} \exists_r}{\forall x(P(x) \to Q(x)) \vdash \exists y(P(a) \to Q(y))} \forall_l$$

$$X : [\{\forall x(P(x) \to Q(x));\}, \{; \exists y(P(a) \to Q(y))\}]$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \qquad Q(a) \vdash Q(a)}{P(a), P(a) \to Q(a) \vdash Q(a)} \to_l}{P(a) \to Q(a) \vdash P(a) \to Q(a)} \to_r}{P(a) \to Q(a) \vdash \exists y (P(a) \to Q(y))} \exists_r}{\forall x (P(x) \to Q(x)) \vdash \exists y (P(a) \to Q(y))} \forall_l
$$

$$X : [\{\forall x (P(x) \to Q(x)); \}, \{; \exists y (P(a) \to Q(y))\}]$$

axioms:

$$[\{; P(a)\}, \{P(a); \}] \text{ and } [\{Q(a); \}, \{; Q(a)\}]$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \qquad Q(a) \vdash Q(a)}{P(a), P(a) \to Q(a) \vdash Q(a)} \to_l}{P(a) \to Q(a) \vdash P(a) \to Q(a)} \to_r}{P(a) \to Q(a) \vdash \exists y(P(a) \to Q(y))} \exists_r}{\forall x(P(x) \to Q(x)) \vdash \exists y(P(a) \to Q(y))} \forall_l$$

$$X : [\{\forall x(P(x) \to Q(x)); \}, \{; \exists y(P(a) \to Q(y))\}]$$

axioms:

$$[\{; P(a)\}, \{P(a); \}] \text{ and } [\{Q(a); \}, \{; Q(a)\}]$$

$\vdash P(a), \neg P(a)$ and $\neg P(a), P(a) \vdash$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \qquad Q(a) \vdash Q(a)}{P(a), P(a) \to Q(a) \vdash Q(a)}\to_l}{P(a) \to Q(a) \vdash P(a) \to Q(a)}\to_r}{P(a) \to Q(a) \vdash \exists y(P(a) \to Q(y))}\exists_r}{\forall x(P(x) \to Q(x)) \vdash \exists y(P(a) \to Q(y))}\forall_l
$$

$$X : [\{\forall x(P(x) \to Q(x)); \}, \{; \exists y(P(a) \to Q(y))\}]$$

axioms:

$$[\{; P(a)\}, \{P(a); \}] \text{ and } [\{Q(a); \}, \{; Q(a)\}]$$

$\vdash P(a), \neg P(a)$ and $\neg P(a), P(a) \vdash$ and $Q(a) \vdash Q(a)$ and $Q(a) \vdash Q(a)$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \qquad Q(a) \vdash Q(a)}{P(a), P(a) \to Q(a) \vdash Q(a)} \to_l}{P(a) \to Q(a) \vdash P(a) \to Q(a)} \to_r}{P(a) \to Q(a) \vdash \exists y(P(a) \to Q(y))} \exists_r}{\forall x(P(x) \to Q(x)) \vdash \exists y(P(a) \to Q(y))} \forall_l$$

$$X : [\{\forall x(P(x) \to Q(x)); \}, \{; \exists y(P(a) \to Q(y))\}]$$

axioms:

$$[\{; P(a)\}, \{P(a); \}] \text{ and } [\{Q(a); \}, \{; Q(a)\}]$$

$$\vdash P(a), \neg P(a) \text{ and } \neg P(a), P(a) \vdash \quad \text{and} \quad Q(a) \vdash Q(a) \text{ and } Q(a) \vdash Q(a)$$

$$\neg P(a) \vee Q(a)$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \qquad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_l}{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow_r}{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l$$

$$X: [\{\forall x(P(x) \rightarrow Q(x)); \}, \{; \exists y(P(a) \rightarrow Q(y))\}]$$

axioms:

$$[\{; P(a)\}, \{P(a); \}] \text{ and } [\{Q(a); \}, \{; Q(a)\}]$$

$\vdash P(a), \neg P(a)$ and $\neg P(a), P(a) \vdash$ and $Q(a) \vdash Q(a)$ and $Q(a) \vdash Q(a)$

$$\neg P(a) \vee Q(a)$$

$$\forall x(\neg P(x) \vee Q(x))$$

The case of atomic cuts

Lemma

Let φ *be an* LK*-proof of the form*

$$\dfrac{\begin{matrix}(\varphi_1) \\ \Gamma \vdash \Delta, F\end{matrix} \qquad \begin{matrix}(\varphi_2) \\ F, \Pi \vdash \Lambda\end{matrix}}{\Gamma, \Pi \vdash \Delta, \Lambda}\ cut$$

F *atomic,*
I *interpolant of* $\Gamma \vdash \Delta, F$*,*
J *interpolant of* $F, \Pi \vdash \Lambda$*.*

Then there exists an interpolant of the end-sequent of the form

$$I \wedge J \quad \text{or} \quad I \vee J.$$

Proof.

Let $X = [\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}]$. We distinguish:

1. F occurs only in $\{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}$. Then,

$$X_1 = [\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2, F\}] \text{ of } \Gamma \vdash \Delta, F$$

$$X_2 = [\{\Pi_1; \Lambda_1\}, \{F, \Pi_2; \Lambda_2\}] \text{ of } F, \Pi \vdash \Lambda.$$

Then there are interpolation derivations $\varphi_1 =$

$$\dfrac{\begin{matrix}(\chi_{1,1}) \\ \Gamma_1 \vdash \Delta_1, I\end{matrix} \qquad \begin{matrix}(\chi_{1,2}) \\ I, \Gamma_2 \vdash \Delta_2, F\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, F}\ cut$$

$\varphi_2 =$

$$\dfrac{\begin{matrix}(\chi_{2,1}) \\ \Pi_1 \vdash \Lambda_1, J\end{matrix} \qquad \begin{matrix}(\chi_{2,2}) \\ J, F, \Pi_2, \vdash \Lambda_2\end{matrix}}{\Pi_1, \Pi_2, F \vdash \Lambda_1, \Lambda_2}\ cut$$

Proof.

Let $X = [\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}]$. We distinguish:

1. F occurs only in $\{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}$. Then,

$$X_1 = [\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2, F\}] \text{ of } \Gamma \vdash \Delta, F$$

$$X_2 = [\{\Pi_1; \Lambda_1\}, \{F, \Pi_2; \Lambda_2\}] \text{ of } F, \Pi \vdash \Lambda.$$

Then there are interpolation derivations $\varphi_1 =$

$$\dfrac{\begin{matrix}(\chi_{1,1})\\ \Gamma_1 \vdash \Delta_1, I\end{matrix} \qquad \begin{matrix}(\chi_{1,2})\\ I, \Gamma_2 \vdash \Delta_2, F\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, F}\ cut$$

$\varphi_2 =$

$$\dfrac{\begin{matrix}(\chi_{2,1})\\ \Pi_1 \vdash \Lambda_1, J\end{matrix} \qquad \begin{matrix}(\chi_{2,2})\\ J, F, \Pi_2, \vdash \Lambda_2\end{matrix}}{\Pi_1, \Pi_2, F \vdash \Lambda_1, \Lambda_2}\ cut$$

Proof.

Let $X = [\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}]$. We distinguish:

1. F occurs only in $\{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}$. Then,

$$X_1 = [\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2, F\}] \text{ of } \Gamma \vdash \Delta, F$$

$$X_2 = [\{\Pi_1; \Lambda_1\}, \{F, \Pi_2; \Lambda_2\}] \text{ of } F, \Pi \vdash \Lambda.$$

$$\dfrac{\dfrac{\dfrac{(\chi_{1,1})}{\Gamma_1 \vdash \Delta_1, I} \quad \dfrac{(\chi_{2,1})}{\Pi_1 \vdash \Lambda_1, J}}{\Gamma_1, \Pi_1 \vdash \Delta_1, \Lambda_1, I \wedge J} \wedge_r \qquad \dfrac{\dfrac{(\chi_{1,2})}{I, \Gamma_2 \vdash \Delta_2, F} \quad \dfrac{(\chi_{2,2})}{J, F, \Pi_2, \vdash \Lambda_2}}{I \wedge J, \Gamma_2, \Pi_2 \vdash \Delta_2, \Lambda_2} \, cut + \wedge_l}{\Gamma_1, \Pi_1, \Gamma_2, \Pi_2 \vdash \Delta_1, \Lambda_1, \Delta_2, \Lambda_2} \, cut$$

Proof.

Let $X = [\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}]$. We distinguish:

2. F occurs only in $\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}$. Then,

$$X_1 = [\{\Gamma_1; \Delta_1, F\}, \{\Gamma_2; \Delta_2\}] \text{ of } \Gamma \vdash \Delta, F$$

$$X_2 = [\{F, \Pi_1; \Lambda_1\}, \{\Pi_2; \Lambda_2\}] \text{ of } F, \Pi \vdash \Lambda.$$

Then there are interpolation derivations $\varphi_1 =$

$$\dfrac{\begin{matrix}(\chi_{1,1}) \\ \Gamma_1 \vdash \Delta_1, F, I\end{matrix} \qquad \begin{matrix}(\chi_{1,2}) \\ I, \Gamma_2 \vdash \Delta_2\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, F}\ cut$$

$\varphi_2 =$

$$\dfrac{\begin{matrix}(\chi_{2,1}) \\ F, \Pi_1 \vdash \Lambda_1, J\end{matrix} \qquad \begin{matrix}(\chi_{2,2}) \\ J, \Pi_2, \vdash \Lambda_2\end{matrix}}{\Pi_1, \Pi_2, F \vdash \Lambda_1, \Lambda_2}\ cut$$

Proof.

Let $X = [\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}]$. We distinguish:

2. F occurs only in $\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}$. Then,

$$X_1 = [\{\Gamma_1; \Delta_1, F\}, \{\Gamma_2; \Delta_2\}] \text{ of } \Gamma \vdash \Delta, F$$

$$X_2 = [\{F, \Pi_1; \Lambda_1\}, \{\Pi_2; \Lambda_2\}] \text{ of } F, \Pi \vdash \Lambda.$$

Then there are interpolation derivations $\varphi_1 =$

$$\dfrac{\dfrac{(\chi_{1,1})}{\Gamma_1 \vdash \Delta_1, F, I} \qquad \dfrac{(\chi_{1,2})}{I, \Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, F} \; cut$$

$\varphi_2 =$

$$\dfrac{\dfrac{(\chi_{2,1})}{F, \Pi_1 \vdash \Lambda_1, J} \qquad \dfrac{(\chi_{2,2})}{J, \Pi_2, \vdash \Lambda_2}}{\Pi_1, \Pi_2, F \vdash \Lambda_1, \Lambda_2} \; cut$$

Proof.

Let $X = [\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}]$. We distinguish:

2. F occurs only in $\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}$. Then,

$$X_1 = [\{\Gamma_1; \Delta_1, F\}, \{\Gamma_2; \Delta_2\}] \text{ of } \Gamma \vdash \Delta, F$$

$$X_2 = [\{F, \Pi_1; \Lambda_1\}, \{\Pi_2; \Lambda_2\}] \text{ of } F, \Pi \vdash \Lambda.$$

$$\dfrac{\dfrac{\dfrac{(\chi_{1,1})}{\Gamma_1 \vdash \Delta_1, F, I} \qquad \dfrac{(\chi_{2,1})}{F, \Pi_1 \vdash \Lambda_1, J}}{\Gamma_1, \Pi_1 \vdash \Delta_1, \Lambda_1, I \vee J}\ \mathit{cut} + \vee_r \qquad \dfrac{\dfrac{(\chi_{1,2})}{I, \Gamma_2 \vdash \Delta_2} \qquad \dfrac{(\chi_{2,2})}{J, \Pi_2, \vdash \Lambda_2}}{I \vee J, \Gamma_2, \Pi_2 \vdash \Delta_2, \Lambda_2}\ \vee_l}{\Gamma_1, \Pi_1, \Gamma_2, \Pi_2 \vdash \Delta_1, \Lambda_1, \Delta_2, \Lambda_2}\ \mathit{cut}$$

Proof.

Let $X = [\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}]$. We distinguish:

3. F does not occur in any of the partitions. Then both constructions from above work.

How about more complex cuts?

Definition

Let φ be an LK-proof of the form

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma \vdash \Delta, F\end{array} \qquad \begin{array}{c}(\varphi_2)\\ F, \Pi \vdash \Lambda\end{array}}{\Gamma, \Pi \vdash \Delta, \Lambda}\ cut$$

F contains predicate symbols $P_1, \ldots, P_k, P_l, \ldots, P_n$.
Arbitrary partition X: $[\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}]$.

F is X-violating if a subset of $P_1, \ldots, P_k$ occurs only in $\Gamma_1, \Pi_1, \Delta_1, \Lambda_1$, and a subset of $P_l, \ldots, P_n$ occurs only in $\Gamma_2, \Pi_2, \Delta_2, \Lambda_2$.

F is X-admissible otherwise.

Lemma

Let φ be an LK*-proof of the form*

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma \vdash \Delta, F\end{array} \qquad \begin{array}{c}(\varphi_2)\\ F, \Pi \vdash \Lambda\end{array}}{\Gamma, \Pi \vdash \Delta, \Lambda}\ cut$$

$X : [\{\Gamma_1, \Pi_1; \Delta_1, \Lambda_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Lambda_2\}]$ *a partition of the end-sequent s.t.* F *is* X*-admissible.*
I *is an interpolant of* $\Gamma \vdash \Delta, F$, J *is an interpolant of* $F, \Pi \vdash \Lambda$*.*

Then there exists an interpolant of S *w.r.t.* X *of the form*

$$I \wedge J \quad \text{or} \quad I \vee J.$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \to Q(u) \vdash Q(u)}\ \to: l}{P(u) \to Q(u) \vdash P(u) \to Q(u)}\ \to: r}{P(u) \to Q(u) \vdash \exists y(P(u) \to Q(y))}\ \exists: r}{\forall x(P(x) \to Q(x)) \vdash \exists y(P(u) \to Q(y))}\ \forall: l}{\forall x(P(x) \to Q(x)) \vdash \forall x \exists y(P(x) \to Q(y))}\ \forall: r \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a), P(a) \to Q(v) \vdash Q(v)}\ \to: l}{P(a) \to Q(v) \vdash P(a) \to Q(v)}\ \to: r}{P(a) \to Q(v) \vdash \exists y(P(a) \to Q(y))}\ \exists: r}{\exists y(P(a) \to Q(y)) \vdash \exists y(P(a) \to Q(y))}\ \exists: l}{\forall x \exists y(P(x) \to Q(y)) \vdash \exists y(P(a) \to Q(y))}\ \forall: l}{\forall x(P(x) \to Q(x)) \vdash \exists y(P(a) \to Q(y))}\ \text{cut}$$

$$X: [\{\forall x(P(x) \to Q(x));\ \}, \{\ ; \exists y(P(a) \to Q(y))\}]$$

Cut-formula $\forall x \exists y(P(x) \to Q(y))$ is X-admissible:

$$[\{\forall x(P(x) \to Q(x)); \forall x \exists y(P(x) \to Q(y))\}, \{\ ;\ \}] \text{ and}$$

$$[\{\forall x \exists y(P(x) \to Q(y));\ \}, \{\ ; \exists y(P(a) \to Q(y))\}]$$

$$I = (\bot \vee \bot) \vee (\neg P(a) \vee Q(v)) = P(a) \to Q(v)$$

$$\Downarrow$$

$$\forall x \exists y(P(x) \to Q(y))$$