Incorrect Proofs and Epsilon Calculus [1]

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Outline

Incorrect Proofs

Euler's Incorrect Basel Problem Proof

Euler became famous by deriving

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1}$$

Let us consider Eulers reasoning.

Euler's Incorrect Basel Problem Proof

Consider the polynomial of even degree

$$b_0 - b_1 x^2 + b_2 x^4 - \ldots + (-1)^n b_n x^{2n} \tag{2}$$

If $b_n = 1$ it has the $2n$ roots $\pm\beta_1, \ldots, \beta_n \neq 0$ then (2) can be written as

$$(x - \beta_1)(x + \beta_1)\ldots(x - \beta_n)(x + \beta_n) \tag{3}$$

$$(-1)^n(\beta_1 - x)(\beta_1 + x)\ldots(\beta_n - x)(\beta_n + x) \tag{4}$$

$$(-1)^n(\beta_1^2 - x^2)\ldots(\beta_n^2 - x^2) \tag{5}$$

where $b_0 = (-1)^n \beta_1^2 \ldots \beta_n^2$

$$b_0 \left(1 - \frac{x^2}{\beta_1^2}\right)\left(1 - \frac{x^2}{\beta_2^2}\right)\ldots\left(1 - \frac{x^2}{\beta_n^2}\right) \tag{6}$$

By comparing coefficients in (2) and (6) one obtains that

$$b_1 = b_0 \left(\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2} + \ldots + \frac{1}{\beta_n^2}\right). \tag{7}$$

Euler's Incorrect Basel Problem Proof

Next Euler considers the Taylor series for $sin(x)$ divided by x

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} \tag{8}$$

which has as roots $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ Now by way of analogy Euler assumes that the infinite degree polynomial (8) behaves in the same way as the finite polynomial (2). Hence in analogy to (6) he obtains

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\ldots \tag{9}$$

and in analogy to (7) he obtains

$$\frac{1}{3!} = \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots\right) \tag{10}$$

which immediately gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Weakest Precondition

What is a Weakest Precondition?

Goal: Given a satisfiable formula B, find the weakest satisfiable A in the language of B such that $A \supset B$ is valid.
Definition: A precondition for a satisfiable formula B is any formula A such that $A \supset B$ is valid. We define order among the satisfiable preconditions as $K < C$, meaning that $K \supset C$ is valid but $C \supset K$ is not valid. A formula A is a *weakest precondition* for B if:

- $A \supset B$ is valid
- A is satisfiable
- A is $<$- minimal

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- A is $<$- minimal

Observation:

- $\perp \supset B$ is always valid, therefore we exclude $\perp$ when searching for weakest preconditions.
- $<$ might not be linear

What is a Weakest Precondition?

Proposition: For a propositional formula A which is satisfiable, weakest preconditions are calculable as minimal conjunctions.

Example

A	**B**	**C**	$(A \vee (B \supset C)) \wedge (A \supset B)$
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	1	1
1	1	0	1

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0	1	1	1
1	0	0	0
1	0	1	0
1	1	1	1
1	1	0	1

Among the weakest preconditions are:

- $(\neg A \wedge \neg B \wedge \neg C)$
- $(\neg A \wedge \neg B \wedge C)$
- $(\neg A \wedge B \wedge C)$
- $(A \wedge B \wedge \neg C)$
- $(A \wedge B \wedge C)$

Failure of Weakest Preconditions in First-Order Logic

First-Order Logic: Quantifiers introduce non-terminating implication chains.

Example

$$\forall x\,(A(p(x)) \supset A(x)) \supset A(0)$$

- $A(0)$, $A(p(0))$, $A(p(p(0)))\ldots$ are all preconditions, but none of them is weakest.
- These preconditions form an infinite descending chain under implication, i.e., each is strictly weaker than the previous one.

Conclusion: Quantification breaks finite precondition minimality.

Change of Paradigm: From Preconditions to Proofs

Instead of considering incorrect expressions, we consider incorrect proofs of correct and incorrect expressions.

From Preconditions to Proofs

Example

$$(\forall x(A(x) \wedge \exists y B(y)) \to \forall x A(x)) \wedge \exists x C(x).$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A(a) \Rightarrow A(a)}{A(a) \wedge \exists y B(y) \Rightarrow A(a)}}{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow A(a)}}{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow \forall x A(x)} \qquad \dfrac{\dfrac{\dfrac{\exists y B(y) \Rightarrow \exists y B(y)}{\exists y B(y) \Rightarrow \exists y C(y)}\ *}{A(a) \wedge \exists y B(y) \Rightarrow \exists y C(y)}}{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow \exists y C(y)}}{\dfrac{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow \forall x A(x) \wedge \exists y C(y)}{\Rightarrow \forall x(A(x) \wedge \exists y B(y)) \to \forall x A(x) \wedge \exists y C(y)}}$$

$\exists x B(x) \to \exists y C(y)$ is precondition, but also $\neg \forall x A(x)$ and $C(a)$ are preconditions.But how to calculate a weakest precondition in general?

Alternative Method

Weakest preconditions may not exist in first-order logic.
Alternative Method: Reduce first-order logic to propositional logic using **Hilbert's ε-calculus**.
Observation:

- We focus on incorrect/incomplete proofs where each quantifier inference is correct;
- Correction is only meaningfulif the proofs can be written in a regular tree-like form.

Epsilon Calculus

The Birth of ε-calculus

- Hilbert & Ackermann (1925-1928) provide (**Foundations of Mathematics**) formal definition of the ε-calculus and use of ε to define quantifiers, formalize quantification via ε-terms and link logic to finitist methods.
- Key tool for formalizing consistency of analysis and finitism.
- Plays a vital role in the foundations of proof theory and formal logic.
- Eliminates the need for quantifiers syntactically.
  Replace $\exists x$ and $\forall x$ with terms like $\varepsilon_x A(x)$ and $\tau_x A(x)$.
- Allows conversion of proofs to quantifier-free form (Herbrand disjunctions).
- Forms a bridge between classical and constructive reasoning.

Reactions and Alternatives

Kolmogorov (1925) **On the Principle of the Excluded Middle** - Critique from the intuitionist perspective. Important for understanding limits of ε.

Skolem (1920) **Logisch-kombinatorische Untersuchungen** - Alternative treatment of quantifiers, related to Skolem functions, which constructs functions directly without ε.

Gödel (1930–31) **Completeness and Incompleteness Theorems** - Places limits on Hilbert's program via metamathematical results (final nail in the coffin of Hilbert's program — why consistency cannot be proven within the system.).

Epsilon Calculus: Basics

David Hilbert (1925-1928) (**Foundations of Mathematics**)

- $\varepsilon_x A(x)$: denotes a witness for x such that $A(x)$ holds if such x exists.

$$\exists x\, A(x) = A(\varepsilon_x A(x)), \quad \forall x\, A(x) = A(\varepsilon_x \neg A(x))$$

Critical Formulas:

$$\text{(introduce } \varepsilon\text{-terms during substitution)} \quad \begin{array}{r} A(t) \to A(\varepsilon_x A(x)) \\ A(\varepsilon_x \neg A(x)) \to A(t) \end{array}$$

Translation Process:

Replace quantifiers using ε-terms recursively

$[A]^\varepsilon \equiv A \quad A$ atomic

$[A \circ B]^\varepsilon \equiv [A]^\varepsilon \circ [B]^\varepsilon \quad \circ \in \{\wedge, \vee, \rightarrow\}$

$[\neg A]^\varepsilon \equiv \neg [A]^\varepsilon$

$[\exists x A(x)]^\varepsilon \equiv [A(x)]^\varepsilon \{\varepsilon_x A(x)/x\}$

$[\forall x A(x)]^\varepsilon \equiv [A(x)]^\varepsilon \{\varepsilon_x \neg A(x)/x\}$

Example

The standard translation of $\exists x A(x) \wedge \forall x B(x)$ is $A(\varepsilon_x A(x)) \wedge B(\varepsilon_x \neg B(x))$.

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ε-expression might become complicated $[\exists x \exists y \exists z A(x,y,z)]^\varepsilon =$

$$
\begin{array}{l}
A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), z)), \\
\varepsilon_y A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), z)), \\
y, \varepsilon_z A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)) \\
, z)), y, z)), \varepsilon_z A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \\
\varepsilon_z A(x, y, z)), z)), \varepsilon_y A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, \\
y, \varepsilon_z A(x, y, z)), z)), y, \varepsilon_z A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \\
\varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), z)), y, z)), z))
\end{array}
$$

Example

The standard translation of $\exists x A(x) \wedge \forall x B(x)$ is $A(\varepsilon_x A(x)) \wedge B(\varepsilon_x \neg B(x))$.

ε-expression might become complicated $[\exists x \exists y \exists z A(x,y,z)]^\varepsilon =$

$$
\begin{aligned}
&A(\varepsilon_x A(x, \varepsilon_y A(x,y,\varepsilon_z A(x,y,z)), \varepsilon_z A(x, \varepsilon_y A(x,y,\varepsilon_z A(x,y,z)), z)),\\
&\varepsilon_y A(\varepsilon_x A(x, \varepsilon_y A(x,y,\varepsilon_z A(x,y,z)), \varepsilon_z A(x, \varepsilon_y A(x,y,\varepsilon_z A(x,y,z)), z)),\\
&y, \varepsilon_z A(\varepsilon_x A(x, \varepsilon_y A(x,y,\varepsilon_z A(x,y,z)), \varepsilon_z A(x, \varepsilon_y A(x,y,\varepsilon_z A(x,y,z))\\
&, z)), y, z)), \varepsilon_z A(\varepsilon_x A(x, \varepsilon_y A(x,y,\varepsilon_z A(x,y,z)), \varepsilon_z A(x, \varepsilon_y A(x,y,\\
&\varepsilon_z A(x,y,z)), z)), \varepsilon_y A(\varepsilon_x A(x, \varepsilon_y A(x,y,\varepsilon_z A(x,y,z)), \varepsilon_z A(x, \varepsilon_y A(x,\\
&y, \varepsilon_z A(x,y,z)), z)), y, \varepsilon_z A(\varepsilon_x A(x, \varepsilon_y A(x,y,\varepsilon_z A(x,y,z)), \varepsilon_z A(x,\\
&\varepsilon_y A(x,y,\varepsilon_z A(x,y,z)), z)), y, z)), z))
\end{aligned}
$$

Now if we abbreviate
$[\exists x \exists y \exists z A(x,y,z)]^\varepsilon \equiv A(e_1(e_2(e_3)), e_2(e_3), e_3)$

$$
\begin{aligned}
e_1(y,z) &\equiv \varepsilon_x A(x,y,z)\\
e_2(z) &\equiv \varepsilon_y A(e_1(y,z), y, z)\\
e_3 &\equiv \varepsilon_z A(e_1(e_2(z)), e_2(z), z)
\end{aligned}
$$

Line-by-line translation of regular proofs into ε-calculus:

- Axiom sequents S become $[S]^{\varepsilon}$
- Structural and propositional rules are promoted unchanged
- $\exists$-right: add $A(t) \to A(\varepsilon_x A(x))$ to antecedent
- $\forall$-left: add $\neg A(t) \to \neg A(\varepsilon_x \neg A(x))$ to antecedent
- $\exists$-left: substitute $a \mapsto \varepsilon_x A(x)$
- $\forall$-right: substitute $a \mapsto \varepsilon_x \neg A(x)$

Conclusion: Valid LK-proofs lead to valid ε-proofs.

Embedding

- Since one can define quantifiers by $\exists x A(x) \equiv A(\varepsilon_x A(x)) \qquad \forall x A(x) \equiv A(\varepsilon_x \neg A(x))$. The usual quantifier rules and axioms can be derived from there.
- This definition embeds first-order logic in the ε-calculus.
- The converse is not true: not every ε/τ- formula is the image of an ordinary quantified formula under this embedding.
- Hence, the ε-calculus is more expressive then predicate calculus, simply because ε-terms can be combined in a more complex ways then quantifiers.

Embedding

Example

$$P(\varepsilon_x \neg P(x,x), \varepsilon_x \neg P(x,x)) \rightarrow P(\varepsilon_x \neg P(x,x), \underbrace{\varepsilon_y P(\varepsilon_x \neg P(x,x), y)}_{e})$$

e has no meaning in first-order logic.

Key Observation:

- In ε-proofs no eigenvariable conditions occur.
- ε-calculus is inductive.
- Unlike usual first-order calculi, the ε-calculus admits an unrestricted Deduction Theorem, therefore a valid ε-proof.
- Therefore, an ε-proof of B from critical formulas

$$A_i(t_{i,j}) \to A(\varepsilon_x A_i(x))$$

is tautology:

$$\bigwedge_{i,j}(A_i(t_{i,j}) \to A(\varepsilon_x A_i(x))) \to B$$

P.S. ε-proof should be structurally sound (if all critical formulas together imply the conclusion, the ε-proof is correct).

- Bounds in ε-proof depend only on the number of critical formulas, not the propositional formulas.

Observation

- This translation works even for ***incorrect or incomplete proofs***, **where quantifier inferences are correct**, making it possible to extract the *minimal information* needed to correct them.
- You can have wrong weak quantifier inferences and you still can translate and extract weakest precondition but you will not be able to apply ε-theorems.

We define LK^* sequent system for incorrect/incomplete proofs:

- In addition to Lk possibly incorrect axioms and rules are allowed of the following form:

$$\Pi \Rightarrow \Gamma \quad \text{(something might be an axiom)}$$

$$\frac{\Pi_1 \Rightarrow \Gamma_1 \qquad \Pi_2 \Rightarrow \Gamma_2}{\Pi \Rightarrow \Gamma}$$

Condition: There should be no incorrect quantifier inference that lead to the wrong critical formulas.

Recall Example

Example

$$\forall x(A(x) \wedge \exists y B(y)) \rightarrow \forall x(A(x) \wedge \exists x C(x)).$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A(a) \Rightarrow A(a)}{A(a) \wedge \exists y B(y) \Rightarrow A(a)}}{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow A(a)}}{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow \forall x A(x)} \quad \dfrac{\dfrac{\dfrac{\exists y B(y) \Rightarrow \exists y B(y)}{\exists y B(y) \Rightarrow \exists y C(y)} \ast}{A(a) \wedge \exists y B(y) \Rightarrow \exists y C(y)}}{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow \exists y C(y)}}{\dfrac{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow \forall x A(x) \wedge \exists y C(y)}{\Rightarrow \forall x(A(x) \wedge \exists y B(y)) \rightarrow \forall x A(x) \wedge \exists y C(y)}}$$

$\exists x B(x) \rightarrow \exists y C(y)$ is precondition, but also $\neg \forall x A(x)$ and $C(a)$ are preconditions.
But how to calculate the weakest precondition in general?

Example

$e_1 \equiv \varepsilon_x \neg A(x)$
$e_2 \equiv \varepsilon_y B(y)$
$e_3 \equiv \varepsilon_y C(y)$
$e_4 \equiv \varepsilon_x \neg (A(x) \wedge B(e_2))$

$$\dfrac{\dfrac{\dfrac{\dfrac{A(e_1) \Rightarrow A(e_1) \qquad \dfrac{B(e_2) \Rightarrow B(e_2)}{B(e_2) \Rightarrow C(e_3)}\,*}{A(e_1) \wedge B(e_2) \Rightarrow A(e_1) \qquad A(e_1) \wedge B(e_2) \Rightarrow C(e_3)}}{A(e_4) \wedge B(e_2) \Rightarrow A(e_1) \qquad A(e_4) \wedge B(e_2) \Rightarrow C(e_3)}}{A(e_4) \wedge B(e_2) \Rightarrow A(e_1) \wedge C(e_3)}}{\Rightarrow A(e_4) \wedge B(e_2) \to A(e_1) \wedge C(e_3)}$$

Example

$e_1 \equiv \varepsilon_x \neg A(x)$
$e_2 \equiv \varepsilon_y B(y)$
$e_3 \equiv \varepsilon_y C(y)$
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$$\dfrac{\dfrac{\dfrac{A(e_1) \Rightarrow A(e_1) \qquad \dfrac{B(e_2) \Rightarrow B(e_2)}{B(e_2) \Rightarrow C(e_3)}*}{A(e_1) \wedge B(e_2) \Rightarrow A(e_1) \qquad A(e_1) \wedge B(e_2) \Rightarrow C(e_3)}}{A(e_4) \wedge B(e_2) \Rightarrow A(e_1) \qquad A(e_4) \wedge B(e_2) \Rightarrow C(e_3)}}{\dfrac{A(e_4) \wedge B(e_2) \Rightarrow A(e_1) \wedge C(e_3)}{\Rightarrow A(e_4) \wedge B(e_2) \rightarrow A(e_1) \wedge C(e_3)}}$$

The ε-proof is

$$A(e_1) \wedge B(e_2) \rightarrow A(e_4) \wedge C(e_3) \Rightarrow A(e_4) \wedge B(e_2) \rightarrow A(e_1) \wedge C(e_3)$$

Note that this retranslation is not trivial and that no general reasonable algorithm is known.

Example

$$(A(e_1) \wedge B(e_2) \rightarrow A(e_4) \wedge C(e_3)) \Rightarrow (A(e_4) \wedge B(e_2) \rightarrow A(e_1) \wedge C(e_3))$$

A_1	B_2	C_3	A_4	$A_1 \wedge B_2 \rightarrow A_4 \wedge C_3$	$A_4 \wedge B_2 \rightarrow A_1 \wedge C_3$	Full Formula
0	0	0	0	1	1	1
0	0	0	1	1	1	1
0	0	1	0	1	1	1
0	0	1	1	1	1	1
0	1	0	0	1	1	1
0	1	0	1	1	0	0
0	1	1	0	1	1	1
0	1	1	1	1	0	0
1	0	0	0	1	1	1
1	0	0	1	1	1	1
1	0	1	0	1	1	1
1	0	1	1	1	1	1
1	1	0	0	0	0	1
1	1	0	1	0	0	1
1	1	1	0	0	0	1
1	1	1	1	1	1	1

$A(e_1) \wedge A(e_4) \wedge B(e_2) \wedge C(e_3)$ $\qquad$ $\forall x A(x) \wedge \forall x (A(x) \wedge \exists y B(y)) \wedge \exists x C(x)$

Epsilon Theorems

Hilbert's Ansatz

We first present the procedure such that if we have several critical formulas belonging to the same ε-term, it allows us to eliminate these critical formulas.
Theorem: Given ε-proof π_ε of the form

$$[A(t_1) \to A(\varepsilon_x A(x)) \wedge \cdots \wedge A(t_n) \to A(\varepsilon_x A(x))] \to B(\varepsilon_x A(x))$$

the resulting Hebrand's disjunction is

$$B(t_1) \vee \cdots \vee B(t_n) \vee B(\varepsilon_x A(x)).$$

$\varepsilon_x A(x)$ can be replaced by one of the t_i that does not contain ε-term. In case all of the t_i contain ε-term, you can replace only by some constant. In such a way we obtain a proof π without ε-term.

Proof.

We use case distinction.

- If $A(t_1)$ holds, then replace $\varepsilon_x A(x)$ everywhere by t_1, then $A(t_1) \to B(t_1)$. Whenever $A(t_2)$ holds, then replace $\varepsilon_x A(x)$ everywhere by t_2 and so on.
- If all $A(t_i)$ are false then it means that $\bigwedge_i \neg A(t_i)$ is true and it implies $B(\varepsilon_x A(x))$.
  So we obtain Herbrand's disjunction
  $B(t_1) \vee \cdots \vee B(t_n) \vee B(\varepsilon_x A(x))$ and if $\varepsilon_x A(x)$ does not occur in one of the $A(t_i)$ we can replace it by t_i.

$\square$

Syntactic Property: Term-replacing preserves the tautological structure of ε-proof.

Example

Simple Case Consider Ansatz with two critical formulas

$$\left.\begin{array}{r} A(t) \to A(\varepsilon_x A(x)) \\ A(s) \to A(\varepsilon_x A(x)) \end{array}\right\} \quad \to B(\varepsilon_x A(x))$$

$$\left.\begin{array}{r} A(t') \to A(t) \\ A(s') \to A(t) \end{array}\right\} \quad \to B(t) \quad \text{gives } A(t) \to B(t)$$

$$\left.\begin{array}{r} A(t'') \to A(s) \\ A(s'') \to A(s) \end{array}\right\} \quad \to B(s) \quad \text{gives } A(s) \to B(s)$$

If both $A(t)$ and $A(s)$ are assumed to be false, then
$\neg A(t) \vee \neg A(s) \to B(t) \vee B(s)$
By applying the rule of excluded middle we get:

$$B(t) \vee B(s) \vee B(\varepsilon_x A(x))$$

Extended First ε-theorem

Theorem

An ε-proof of the translation of the existential formula can be stepwise transformed into a Herbrand disjunction.

Proof.

Hilbert's Ansatz is repeatedly applied to ε-terms of maximal rank (**soundness**) and maximal degree (**termination**) and to the remaining critical formulas. □

1. **Rank** - the length of the overbinding chain (subordination)
2. **Degree** - the depth of the nesting of ε-terms (inclusion)

Remark: Theorem is only suitable for derived translations of existential formulas. Other quantifiers can be replaced by Skolem functions.

Second epsilon theorem

Theorem (Second ε-theorem)

If A is a Herbrand disjunction of an existential formula after skolemization we obtain a proof of the original prenex formula.

- Skolemization is necessary to transform arbitrary formulas into existential formulas. The second ε-theorem allows to reconstruct proof of the original formulas from Herbrand's disjunction for the prenex formulas.
- Theorem holds for infix formulas: M. Baaz, S. Hetzl, and D. Weller. **On the complexity of proof deskolemization.** The Journal of Symbolic Logic, 77(2):669–686, 2012.
- The connection of the Second ε-theorem to deSkolemization procedure: M. Baaz, M. Gamsakhurdia, R. Iemhoff and R. Jalali. **Skolemization In Intermediate Logics.** (Preprint 2025)

Flow of Errors via Epsilon Calculus

Flow of Errors

Theorem

Let an incorrect ε-proof of ε-expression $E(\bar{e})$ with (weakest) correcting preconditions $F(\bar{e})$ be given. The Herbrand disjuction $\bigvee E(\bar{t}_i)$ is either valid or the $\bigwedge F(\bar{t}_i)$ is a correcting precondition. The bounds are the same as for the Herbrand disjunction.

Corollary

let A be an ε-free sentence which corrects the ε-proof then A corrects also any Herbrand disjunction calculated by the extended first ε-theorem.

Theorem

- *Any ε-free propositional formula P which is implied by ε-proof, is also implied by the Herbrand disjunction obtained by the Extended first ε-theorem.*
- *Any ε-free propositional formula P equivalent to an ε-proof is also equivalent to Herbrand disjunction obtained by the Extended first ε-theorem.*
- *If disjunction of weakest preconditions $\vee F(t_i)$ which is equivalent to some ε-free propositional formula P, is equivalent to ε-proof, then it is also equivalent to its Herbrand disjunction.*

Predicative ε-proofs

Predicative ε-proofs

Definition

An ε-proof is *predicative* iff there is at least one elimination sequence of the critical formulas $A(t) \to A(\varepsilon_x A(x))$, where $\varepsilon_x A(x)$ does not occur in t at the moment of evaluation.

Proposition: Every formula $E(\bar{e})$ which is ε-translation of an existential formula admits a predicative ε-proof.

Proof.

Calculate the Herbrand disjunction $\bigvee_i E(\bar{t}_i)$ of a non-predicative ε-proof. Recover stepwise the critical formulas from Herbrand disjunction and contract. $\square$

Predicative ε-proofs

Example

$$\bigwedge_i (A(t_i) \supset A(\varepsilon_x A_i(x))) \supset E(\bar{e}))$$

$$\Downarrow$$

$$E(s) \vee E(t)$$

$$\Downarrow$$

$$(A(s) \supset A(\varepsilon_x A(x)) \wedge A(t) \supset A(\varepsilon_x A(x))) \supset E(\bar{e})$$

Critical Formulas

Number theories can be expressed using two types of critical formulas

- **Standard critical formulas:**

$$A(t) \to A(\varepsilon_x A(x))$$

- **Inductive critical formulas:**

$$A(t) \to A(\varepsilon_x A(x) \leq t)$$

False-Tolerance

False-Tolerance

Theorem

The algorithm of the extended first ε-theorem is false-tolerant: If an ε-proof fails under exactly one interpretation (one line in its minimal truth table), then its extracted Herbrand disjunction also fails under exactly one interpretation.

- Preserves the degree of failure.
- Resulting Herbrand disjunction may be syntactically longer.
- The Herbrand disjunction is not necessarily incorrect when the ε-proof is incorrect, it may actually be correct:

Example

$(A(t) \supset A(e)) \supset (A(e) \supset A(t))$ with t not containing e, is not correct, but its Herbrand disjunction $A(t) \supset A(t)$ is correct.

Summary

- ε-calculus: remove quantifiers via witnesses.
- False-tolerance: preserves the degree of errors.
- Applications: incorrect/incomplete poofs in **mathematics** and Ratio Decidendi in **English Common Law.**

Conclusion

Most relevant question: what is the retranslation of an ε -expressions to first-order expressions?

$$P(\varepsilon_x \neg P(x,x), \varepsilon_x \neg P(x,x)) \to P(\varepsilon_x \neg P(x,x), \underbrace{\varepsilon_y P(\varepsilon_x \neg P(x,x), y)}_{e})$$

e has no meaning in first-order logic. Any satisfying answer implies the solution to the **problem of T. Arai**: Is there an elementary retranslation of LK^{ε}-proofs with cuts in LK-proofs with cuts?

Further Reading

- Avigad, J. and Zach, R. ”The Epsilon Calculus,” *Stanford Encyclopedia of Philosophy*, 2013. `https://plato.stanford.edu/entries/epsilon-calculus/`
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Q&A

Thank you!