Decidability of the Bernays–Schönfinkel Class of Gödel Logics[1]

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Outline

1. Gödel logics
2. Semantic Core: Skolemization
3. BS class
4. 1-satisfibility: Gluing Lemma
5. Conclusion

Motivation

- Classical first-order logic: every formula can be written in **prenex** form, satisfiability/validity is not decidable for prenex fragments.
- **Bernays–Schönfinkel (BS) class** (1928): class of function-free, quantifier prefix sentences with prefixes $\exists\bar{x}\forall\bar{y}A(\bar{x},\bar{y})$ (satisfiability) and $\forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y})$ (validity).
- **Classical & Intuitionistic** logic: BS class is decidable.
- Question: **Is the BS class decidable in Gödel logics?**

History of Gödel Logics

- **1932–33 Gödel:** Introduced intermediate logics G_n. Proved existence of infinitely many logics between classical and intuitionistic.
- **1959 – Dummett's Infinite-Valued Gödel Logic:** Replaced finite chains by the full $[0, 1]$ interval, adding the linearity axiom $(A \to B) \vee (B \to A)$;
- **1991 – Avron's Hypersequent Calculus** with a communication rule capturing linearity proof-theoretically.
- **1998 – Hájek's t-Norm Logics:** Unified Gödel, Łukasiewicz, and Product logics into the t-norm–based fuzzy-logic framework.
- **Since 1990s – Viennese School:** Baaz, Ciabattoni, Metcalfe, Olivetti, Pichler, Zach *et al.* deepened proof theory, fragments, Kripke semantics, and complexity analysis.
- And many more...

First-order Gödel Logics

- Many-valued logics G_V with $V \subseteq [0,1]$, containing 0 and 1.
- Evaluation ($\neg A = A \to \bot$):

$$
\begin{array}{ll}
(1) & \mathcal{I}(\bot) = 0 \\
(2) & \mathcal{I}(A \wedge B) = min\{\mathcal{I}(A), \mathcal{I}(B)\} \\
(3) & \mathcal{I}(A \vee B) = max\{\mathcal{I}(A), \mathcal{I}(B)\} \\
(4) & \mathcal{I}(A \supset B) = \begin{cases} \mathcal{I}(B) & \text{if } \mathcal{I}(A) > \mathcal{I}(B), \\ 1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B). \end{cases} \\
(5) & \mathcal{I}(\forall x A(x)) = \inf\{\mathcal{I}(A(u))\ u \in U_{\mathcal{I}}\} \\
(6) & \mathcal{I}(\exists x A(x)) = \sup\{\mathcal{I}(A(u))\ u \in U_{\mathcal{I}}\}
\end{array}
$$

$$
\mathcal{I}(\triangle A) = \begin{cases} 1 & \text{if } \mathcal{I}(A) = 1, \\ 0 & \text{otherwise} \end{cases} \qquad \text{Absoluteness operator}
$$

Negation

This yields the following definition of the semantics of $\neg$:

$$\mathcal{I}(\neg A) = \begin{cases} 0 & \text{if } \mathcal{I}(A) > 0 \\ 1 & \text{otherwise} \end{cases}$$

Takeuti's observation

Gödel implication

$$\mathcal{I}(A \to B) = \begin{cases} \mathcal{I}(B) & \text{if } \mathcal{I}(A) > \mathcal{I}(B) \\ 1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B). \end{cases}$$

is the only one satisfying:

- $\mathcal{I}(A) \leq \mathcal{I}(B) \Leftrightarrow \mathcal{I}(A \to B) = 1$
- $\Pi \cup \{A\} \models B \Leftrightarrow \Pi \models A \to B$
- $\Pi \models B \Rightarrow \min\{\mathcal{I}(A) : A \in \Pi\} \leq \mathcal{I}(B)$
  (and if $\Pi = \emptyset \Rightarrow 1 \leq \mathcal{I}(B)$)

Key Concepts

Definition (1-entailment)

For a truth value set V, a (possibly infinite) set Γ of formulas (1-)entails a formula A if the interpretation $\mathcal{I}$ on V of A is 1 in case the interpretations of all formulas in Γ are 1, i.e.,
$\Gamma \Vdash_V A \Longleftrightarrow (\forall \mathcal{I}, \forall B \in \Gamma : \mathcal{I}(B) = 1) \rightarrow \mathcal{I}(A) = 1.$

- Validity: formula evaluates to 1 in all interpretations.
- 1-satisfiability: some interpretation assigns 1.
- Validity and unsatisfiability are **not dual** in Gödel logic, e.g. $A \vee \neg A$ is not valid but its negation is unsatisfiable.
- Depends only on the relative ordering and the topological type of the truth set, and not on their specific values.

Definition of the logic

$$\mathbf{G}_V = \{A : \forall v \text{ into } V : v(A) = 1\}$$

Examples

$V = \{0, 1\}$ $\rightarrow \mathbf{G}_V = CPL$

$V_1 = \{0, 1/2, 1\}, V_2 = \{0, 1/3, 1\}$ $\rightarrow \mathbf{G}_{V_1} = \mathbf{G}_{V_2}$

$V_\uparrow = \{1 - 1/n : n \geq 1\} \cup \{1\}$ $\rightarrow \mathbf{G}_{V_\uparrow} = \mathbf{G}_\uparrow$

$V_\downarrow = \{1/n : n \geq 1\} \cup \{0\}$ $\rightarrow \mathbf{G}_{V_\downarrow} = \mathbf{G}_\downarrow$

$V_m = \{1\} \cup \{1 - 1/k : 1 \geq k \geq m - 1\}$ $\rightarrow \mathbf{G}_{V_m} = \mathbf{G}_m$

Why Define a Logic by Its Valid Sentences?

Issue: Different value-sets $V \subseteq [0,1]$ can induce the *same* Gödel logic G_V.

- e.g.

  $$V_\uparrow = \{1 - \tfrac{1}{n} : n \geq 1\} \cup \{1\} \quad \text{and} \quad V'_\uparrow = \{1 - \tfrac{2}{n} : n \geq 1\} \cup \{1\}$$

  both yield $G_\uparrow$.
- In fact, there are *uncountably many* closed subsets of $[0,1]$ but only *countably many* distinct propositional Gödel logics.
- This ensures a *one-to-one correspondence* between:

  $$\{\text{distinct Gödel logics}\} \longleftrightarrow \{\text{sets of formulas } G_V\}.$$

First-Order Gödel logics

$$V_{[0,1]} = [0,1] \longrightarrow \mathrm{G}_{[0,1]}$$
$$V_{\downarrow} = \{0\} \cup \{1/k : k \geq 1\} \longrightarrow G_{\downarrow},$$
$$V_{\uparrow} = \{1\} \cup \{1 - 1/k : k \geq 1\} \longrightarrow G_{\uparrow}$$
$$V_m = \{1\} \cup \{1 - 1/k : 1 \geq k \geq m-1\} \longrightarrow G_m$$

$$(Lin) \quad (A \to B) \vee (B \to A)$$
$$(CD) \quad \forall x(A(x) \vee B) \supset (\forall x A(x) \vee B)$$
$$(Iso_0) \quad \forall x \neg\neg A(x) \supset \neg\neg \forall x A(x)$$
$$(Iso_1) \quad \Delta \exists x A(x) \to \exists x \Delta A(x)$$
$$(Fin) \quad (\top \supset A_1) \vee (A_1 \supset A_2) \vee \cdots \vee (A_{m-1} \supset \bot)$$

Relationships between first-order Gödel logics

Proposition 1

Whenever $V \subseteq V'$ then $G_{V'} \subseteq G_V$.

Proposition 2 (Baaz,Leitsch, Zach).

The following containment relationships hold:

1) $G_m \supsetneq G_{m+1}$
2) $G_m \supsetneq G_\uparrow \supsetneq G_{[0,1]}$
3) $G_m \supsetneq G_\downarrow \supsetneq G_{[0,1]}$
4) $G_{[0,1]} = \bigcap_V G_V$
5) $G_\uparrow = \bigcap_{m \geq 2} G_m$.

Prenex Fragments

Prenex Fragments (Logically Equivalence)

Logically Equivalent Prenex Normal Forms				
	without $\triangle$		with $\triangle$	
Gödel set V	1-valid	$>$ 0-valid	1-valid	$>$ 0-valid
finite	✓	✓	✓	✓
$G_{\uparrow}$	✓	✓	×	×
Count. without $G_{\uparrow}$	×	×	×	×
0 isolated	×	✓	×	×
0 not isolated	×	×	×	×

Prenexation fails for $G_{[0,1]}$ when 0 is not isolated:

$$\neg\forall x A(x) \wedge \forall x \neg\neg A(x)$$

The same holds for all Gödel logics where there is only one cumulation point from above.

Possible truth value sets

Perfect set

A set $P \subseteq \mathbb{R}$ is perfect if it is closed and all its points are limit points in P.

Cantor-Bendixon

Any closed $V \subseteq \mathbb{R}$ can be uniquely written as $V = P \cup C$, with P a perfect subset of V and C countable and open.

Examples for perfect sets

- $[0, 1]$, any closed interval, any finite union of closed intervals
- Cantor Middle Third set $\mathbb{C}$: all numbers of $[0, 1]$ that do not have a 1 in the triadic notation (cut out all open middle intervals recursively) (perfect but nowhere dense)

Full characterization of Axiomatizability

Theorem (Trakhtenbrot, 1950)

The set of first-order sentences that are valid in all finite models is not recursively enumerable.

Similarly, the validity in first-order Gödel logic is characterized by the following theorem.

Theorem (Baaz, Preining)

A first-order Gödel logic G_V is recursively enumerable iff one of the following conditions is satisfied:

1. V is finite,

2. V is uncountable and 0 is an isolated point,

3. V is uncountable, and every neighbourhood of 0 is in the prefect subset.

Incompleteness of First-order Goedel Logics

Not recursively enumerable

- countably infinite truth value set
- every neighbourhood of 0 is countably infinite

(Preining – PhD; Baaz, Preining, Zach 2007)

Prenex Fragments (Validity Equivalence)

Validity Equivalent Prenex Normal Forms			
Gödel set V		without $\triangle$	with $\triangle$
Finite		$\checkmark$	$\checkmark$
Uncountable	0 isolated	$\checkmark$	$\checkmark$
	0 is in perfect set	$\checkmark$	$\checkmark$
	0 not in perfect set	$\times$	$\times$
Countable		Open	

- Prenex fragment in the uncountable case is always r. e. as derivability in $\mathbf{G}_{[0,1]}$ can be expressed by Kleene's T (putting double negation in front of all atoms and shifting the quantifiers in the classical way).
- Prenex fragments without references to 0 ($\perp$-free) are r.e. iff it is uncountable

Theorem

The prenex fragment of $\mathbf{G}_V$ is r.e. if and only if V is finite or uncountable. The prenex fragments of any two $\mathbf{G}_V$ where V is uncountable coincide.

Semantic Core: Skolemization

Skolemization

- Skolemization enables decidability in BS class.
- Validity and 1-satisfiability in BS class are decidable for all Gödel logics

Lemma (Skolemization preserves validity)

For all prenex formulas $Q\bar{x}A(\bar{x})$ and all Gödel logics G

$$\Gamma \Vdash_G Q\bar{x}A(\bar{x}) \iff \Gamma \Vdash_G (Q\bar{x}A(\bar{x}))^S$$

where $Q\bar{x}$ is a quantifier prefix and $A(\bar{x})$ is a quantifier-free formula.

Proof.

It is sufficient to prove with A arbitrary and f a new function:

$$\Gamma \Vdash_G \exists \overline{x} \forall y A(\overline{x}, y) \Leftrightarrow \Gamma \Vdash_G \exists \overline{x} A(\overline{x}, f(\overline{x})).$$

It follows then from induction. $(\Rightarrow)$ The direction from left to right is obvious.
$(\Leftarrow)$ For the other direction, if $\nVdash_G \exists \overline{x} \forall y A(\overline{x}, y)$ then for some interpretation $\mathcal{I}$

$$\sup\{d_{\overline{c}} \mid \mathcal{I}(\forall y A(\overline{c}, y)) = d_{\overline{c}}\} \leq d < 1.$$

Using the axiom of choice we can assign a value for every $f(\overline{c})$ such that $\mathcal{I}(A(\overline{c}, f(\overline{c}))$ is in between $d_{\overline{c}}$ and $d_{\overline{c}} + \frac{1-d}{2}$. As a consequence

$$\sup\{d_{\overline{c}} + \frac{1-d}{2} \mid \mathcal{I}(A(\overline{c}, f(\overline{c}))) \leq d_{\overline{c}} + \frac{1-d}{2}\} \leq d + \frac{1-d}{2} < 1$$

and thus $\Gamma \nVdash_G \exists \overline{x} A(\overline{x}, f(\overline{x}))$. $\square$

BS class

Theorem

Validity in Berneys-Schönfinkel (BS) class is decidable for all Gödel logics.

Proof.

from above lemma follows

- Key transformation: $\forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y}) \rightsquigarrow \exists\bar{y}A(\bar{c},\bar{y})$

$$\Vdash_G \forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y}) \Longleftrightarrow \Vdash_G \exists\bar{y}A(\bar{c},\bar{y})$$

  for new constants $\bar{c}$. (This is the core - "Skolemization step" - validity reduces to existential formulas with constants only.)
- After this, validity becomes a decidable problem for BS class.
- Use constant domain countermodels to check validity - Suppose there is a countermodel M such that $M \nVdash_G \exists\bar{y}A(\bar{c},\bar{y})$. Then there is also a countermodel M' such that $M' \nVdash_G \exists\bar{y}A(\bar{c},\bar{y})$ where the domain of M' contains only interpretations of $\bar{c}$.

□

Corollary

1) *Let $\exists\bar{y}A(\bar{y})$ contain only constants $\bar{c}$, then Herbrand's theorem holds for $\exists\bar{y}A(\bar{y})$ for all Gödel logics G.*
2) *Let $\forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y})$ prenex formulas contain only constants $\bar{d}$, then $\Vdash_G \forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y}) \Longleftrightarrow \Vdash_{G'} \forall\bar{x}\exists\bar{y}A(\bar{x},\bar{y})$ for all infinitely-valued Gödel logics G, G'.*

Proof.

1) According to the proof of the above theorem, $M \nVdash_G \exists\bar{y}A(\bar{c},\bar{y})$ implies $M' \nVdash_G \exists\bar{y}A(\bar{c},\bar{y})$ with restricted domain to constants only.
2) follows from 1), as Herbrand disjunction is contained in $\bigvee_n A(\bar{c}_n,\bar{d}_n)$ where $\bar{c}_n,\bar{d}_n$ are possible variations of $\bar{c},\bar{d}$ and validity for propositional formulas coincides with infinitely-valued Gödel logics. □

Remark

Note that 1) is not trivial as prenex formulas and consequently $\exists$-formulas (see. Skolemization Lemma) for countable Gödel logics are not r.e.

1-satisfibility: Gluing Lemma

Lemma (Gluing lemma)

Let $\mathcal{I}$ be an interpretation into $V \subseteq [0,1]$. Let us fix a value $\omega \in [0,1]$ and define

$$\mathcal{I}_\omega(\mathcal{P}) = \begin{cases} \mathcal{I}(\mathcal{P}) & \text{if } \mathcal{I}(\mathcal{P}) \leq \omega, \\ 1 & \text{otherwise} \end{cases}$$

for atomic formula $\mathcal{P}$ in $\mathcal{L}^{\mathcal{I}}$. Then $\mathcal{I}_\omega$ is an interpretation into V such that

$$\mathcal{I}_\omega(\mathcal{B}) = \begin{cases} \mathcal{I}(\mathcal{B}) & \text{if } \mathcal{I}(\mathcal{B}) \leq \omega, \\ 1 & \text{otherwise} \end{cases}$$

Theorem

1-satisfiability in first-order Gödel logics coincides with classical satisfiability iff 0 is isolated.

Proposition

In the following cases, 1-satisfiability in Gödel logics is classical satisfiability:

1) In the propositional case

2) In the first-order case, where the truth value set is arbitrary but 0 is isolated

3) The prenex fragment for any truth value set

4) The existential fragments for any truth value set

5) The $\perp$ - free fragment is $G_{[0,1]}$ 1-satisfiable iff it is classical satisfiable.

Theorem

1-satisfiability in Berneys-Schönfinkel class is decidable for all Gödel logics.

Proof.

The proof is obvious as 1-satisfiability coincides with classical satisfiability and, therefore, is decidable. □

Corollary

1-satisfiability of monadic fragments is always decidable if 0 is isolated.

Remark

- *All Gödel logics coincide for the BS class w.r.t. 1-satisfiability, but only the infinitely valued Gödel logics coincide for the BS class w.r.t. to validity.*
- *The BS fragment of any infinitely-valued Gödel logic is the intersection of the BS fragments of the finitely-valued Gödel logic, both for satisfiability and validity.*

Conclusion

Fragment	Skolemizable	r.e.	Decidable
First-order Gödel Logic	no	no	no
Prenex	yes	sometimes	not known
BS Class	yes	yes	yes
Monadic finite	not known	yes	yes
Monadic infinite	not known	not known	no

Conclusion

Fragment	Skolemizable	r.e.	Decidable
First-order Gödel Logic	no	no	no
Prenex	yes	sometimes	not known
BS Class	yes	yes	yes
Monadic finite	not known	yes	yes
Monadic infinite	not known	not known	no

- All prenex intermediate logics admit Skolemization and BS class of these logics are decidable.
- Ackermann class is also decidable.
- Study *other* fragments: existential, monadic ...
- What happens when $\triangle$ is present.
- Study decidable classes of non R.E. Gödel logics as for $G_{\uparrow}$ is the intersection of all finite Gödel logics and $G_{\downarrow}$ relates to temporal logics.

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Q&A

Thank you!