# Cut-elimination for non-wellfounded sequent calculi for IL

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# Introduction: Interpretability logic $\mathbf{IL}$

- **interpretability logic**: a modal logic corresponding to the notion of relative interpretability between first-order arithmetical theories
- **syntax of interpretability logic**: given by

$$\phi ::= p \mid \bot \mid \phi \to \phi \mid \phi \rhd \phi,$$

  where $p$ ranges over a fixed set of propositional variables
- we treat other Boolean connectives and modal operators $\Box$ and $\Diamond$ as abbreviations:

$$\neg\phi := \phi \to \bot, \qquad \top := \neg\bot, \qquad \phi \wedge \psi := \neg(\phi \to \neg\psi),$$
$$\phi \vee \psi := \neg\phi \to \psi, \qquad \Box\phi := \neg\phi \rhd \bot, \qquad \Diamond\phi := \neg(\phi \rhd \bot).$$

# Introduction: Interpretability logic IL

The Hilbert-style axiomatization of $\mathrm{IL}$ is given by the following axioms:

- tautologies of classical propositional logic,

(K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$,

(4) $\Box\phi \to \Box\Box\phi$,

(L) $\Box(\Box\phi \to \phi) \to \Box\phi$,

(J1) $\Box(\phi \to \psi) \to (\phi \rhd \psi)$,

(J2) $(\phi \rhd \chi) \land (\chi \rhd \psi) \to (\phi \rhd \psi)$,

(J3) $(\phi \rhd \psi) \land (\chi \rhd \psi) \to (\phi \lor \chi) \rhd \psi$,

(J4) $\phi \rhd \psi \to (\Diamond\phi \to \Diamond\psi)$,

(J5) $\Diamond\phi \rhd \phi$,

and inference rules modus ponens and necessitation:

$$\frac{\phi \to \psi \qquad \phi}{\psi}\,, \qquad \frac{\phi}{\Box\phi}\,.$$

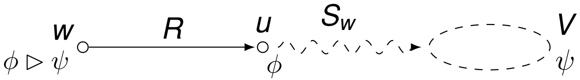
# Generalised Veltman (Verbrugge) semantics

- basic semantics for interpretability logic: **Veltman semantics**
- **Verbrugge semantics** is a generalization of Veltman semantics
- an ordered quadruple $(W, R, \{S_w : w \in W\}, \Vdash)$ is called a **Verbrugge model** if it satisfies the following conditions:
  (i) $W$ is a non-empty set and $R \subseteq W \times W$ is a transitive and reverse well-founded
  (ii) $S_w \subseteq R[w] \times (\mathcal{P}(R[w]) \setminus \{\emptyset\})$
  (iii) $wRu$ implies $uS_w\{u\}$
  (iv) If $uS_wV$ and $(\forall v \in V)(vS_wZ_v)$, then $uS_w\left(\bigcup_{v\in V} Z_v\right)$
  (v) If $wRuRv$ then $uS_w\{v\}$
  (vi) If $uS_wV$ and $V \subseteq Z \subseteq R[w]$ then $uS_wZ$

# Verbrugge semantics

(vii) $\Vdash$ is a forcing relation defined as usual in Boolean cases, and

$$w \Vdash \phi \rhd \psi \text{ iff } \forall u((wRu \;\&\; u \Vdash \phi) \Rightarrow \exists V(uS_w V \& (\forall v \in V)(v \Vdash \psi))).$$



The current state of research regarding the principles that are used to extend $\mathrm{IL}$:

| | principle | compl(o) | compl(g) | FMP(o) | FMP(g) |
|---|---|---|---|---|---|
| M | $\phi \rhd \psi \to \phi \wedge \Box\xi \rhd \psi \wedge \Box\xi$ | ✓ | ✓ | ✓ | ✓ |
| $\mathrm{M}_0$ | $\phi \rhd \psi \to \Diamond\phi \wedge \Box\xi \rhd \psi \wedge \Box\xi$ | ✓ | ✓ | ? | ✓ |
| P | $\phi \rhd \psi \to \Box(\phi \rhd \psi)$ | ✓ | ✓ | ✓ | ✓ |
| $\mathrm{P}_0$ | $\phi \rhd \Diamond\psi \to \Box(\phi \rhd \psi)$ | ✗ | ✓ | ? | ✓ |
| R | $\phi \rhd \psi \to \neg(\phi \rhd \neg\xi) \rhd \psi \wedge \Box\xi$ | ? | ✓ | ? | ✓ |
| W | $\phi \rhd \psi \to \phi \rhd \psi \wedge \Box\neg\phi$ | ✓ | ✓ | ✓ | ✓ |
| F | $\phi \rhd \Diamond\phi \to \Box\neg\phi$ | ✗ | ? | ✓ | ✓ |
| $\mathrm{W}^*$ | $\phi \rhd \psi \to \psi \wedge \Box\xi \rhd \psi \wedge \Box\xi \wedge \Box\neg\phi$ | ✓ | ✓ | ? | ✓ |

- for example, the logic $\mathrm{ILP}$ is obtained from $\mathrm{IL}$ by adding the persistence principle $\mathrm{P}$ as an axiom

# The systems of sequent calculus for interpretability logics

- cut-free systems for logics $\mathrm{IL}$ and $\mathrm{IK4}$ (axioms of $\mathrm{IL}$ without the $\mathrm{L}$ axiom):

  Katsumi Sasaki, *A cut-free sequent system for the smallest interpretability logic*, Studia Logica, 70(3):353–372, 2002.

- another cut-free system for the logic $\mathrm{IK4}$:

  Katsumi Sasaki, *A sequent system for a sublogic of the smallest interpretability logic*, Journal of the Nanzan Academic Society, Mathematical Sciences and Information Engineering, 3:1–12, 2003.

- cut-free system for logic $\mathrm{IK4P}$ and system for $\mathrm{ILP}$ (that is conjectured to be cut-free):

  Katsumi Sasaki, *A sequent system for the interpretability logic with the persistence axiom*, Journal of the Nanzan Academic Society, Mathematical Sciences and Information Engineering, 2:25–34, 2002.

# An overview of the results that will be presented here

$$
\begin{tikzcd}
\mathrm{IL} \arrow[r, shift left] & \mathrm{GIL} + \mathrm{Cut} \arrow[l, shift left] \arrow[r] & \mathrm{G}^{\infty}\mathrm{IL} + \mathrm{Cut} \arrow[d] \\
 & \mathrm{GIL} \arrow[u, "\text{trivial}"'] & \mathrm{G}^{\infty}\mathrm{IL} \arrow[l]
\end{tikzcd}
$$

# Sequents

- a **sequent** is an expression of the form

$$\Gamma \Rightarrow \Delta$$

  where $\Gamma$ and $\Delta$ are finite multisets of $(\mathrm{IL}$-)formulas
- given a multiset of formulae $\Phi$, we use the following notation:

$$\Phi \rhd \bot := \{\phi \rhd \bot \mid \phi \in \Phi\}$$

$$\Phi_X := \{\phi_i \mid i \in X\}$$

- we write $[S_i]_{m \ldots i \ldots 0}$ for the finite sequence of sequents $(S_m, S_{m-1}, \ldots, S_0)$
- also we write $\phi_0, \ldots, \phi_k, \Gamma_0, \ldots, \Gamma_l \Rightarrow \Delta_0, \ldots, \Delta_m, \psi_0, \ldots, \psi_n$ instead of $\{\phi_0, \ldots, \phi_k\} \cup \Gamma_0 \cup \cdots \cup \Gamma_l \Rightarrow \Delta_0 \cup \cdots \cup \Delta_m \cup \{\psi_0, \ldots, \psi_n\}$

# Wellfounded sequent calculus GIL

We define the sequent calculus $\mathrm{GIL}$ as the wellfounded calculus given by the following rules:

$$\frac{}{p, \Gamma \Rightarrow p, \Delta}\ \mathsf{ax} \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\ \bot\mathsf{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \to \psi, \Gamma \Rightarrow \Delta} \to \mathsf{L} \qquad \frac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \to \psi} \to \mathsf{R}$$

$$\frac{[\psi_i \rhd \bot, \psi_i, \Phi_{[0,i)} \rhd \bot, \phi \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi]_{m \ldots i \ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta} \rhd_{\mathrm{IL}}$$

The system $\mathrm{GIL} + \mathrm{Cut}$ also has the following $\mathrm{Cut}$ rule:

$$\frac{\Gamma \Rightarrow \Delta, \chi \qquad \chi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{Cut}$$

# The rule $\rhd_{\mathrm{IL}}$ - an example

Let $m = 2$. Then we have

$$\dfrac{S_2 \qquad S_1 \qquad S_0}{\phi_0 \rhd \psi_0, \phi_1 \rhd \psi_1, \Gamma \Rightarrow \psi_2 \rhd \phi, \Delta} \rhd_{\mathrm{IL}}$$

where

$$S_2 = \psi_2 \rhd \bot, \psi_2, \phi_0 \rhd \bot, \phi_1 \rhd \bot, \phi \rhd \bot \Rightarrow \phi_0, \phi_1, \phi$$
$$S_1 = \psi_1 \rhd \bot, \psi_1, \phi_0 \rhd \bot, \phi \rhd \bot \Rightarrow \phi_0, \phi$$
$$S_0 = \psi_0 \rhd \bot, \psi_0, \phi \rhd \bot \Rightarrow \phi$$

# An example of a wellfounded proof in $\mathrm{GIL}$

A wellfounded **proof** in $\mathrm{GIL}$ (+ $\mathrm{Cut}$) is a finite tree whose nodes are marked by sequents and whose leaves are marked by initial sequents (ax, $\bot$L) and that is constructed to the rules of $\mathrm{GIL}$ (+ $\mathrm{Cut}$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\neg p \rhd \bot, p, \bot \rhd \bot, p \rhd \bot \Rightarrow \bot, p}\,\mathrm{ax}}{\neg\neg p \rhd \bot, \bot \rhd \bot, p \rhd \bot \Rightarrow \bot, p, \neg p}\,\neg\mathrm{R}}{\neg\neg p \rhd \bot, \neg\neg p, \bot \rhd \bot, p \rhd \bot \Rightarrow \bot, p}\,\neg\mathrm{L} \qquad \dfrac{}{\bot \rhd \bot, \bot, \bot \rhd \bot \Rightarrow \bot}\,\bot\mathrm{L}}{\Diamond p \rhd \bot, p \rhd \bot \Rightarrow p, \neg\neg p \rhd \bot}\,\rhd_{\mathrm{IL}}}{\Diamond p \rhd \bot, \Diamond p, p \rhd \bot \Rightarrow p}\,\neg\mathrm{L}}{\Rightarrow \Diamond p \rhd p}\,\rhd_{\mathrm{IL}}$$

So we have proved that $\mathrm{GIL} \vdash \Diamond p \rhd p$.

# Hilbert style proofs in $\mathrm{IL}$ and sequent proofs in $\mathrm{GIL} + \mathrm{Cut}$

- By induction on the length of the Hilbert proof of $\phi$ we can prove:

  if $\mathrm{IL} \vdash \phi$, then $\mathrm{GIL} + \mathrm{Cut} \vdash \Rightarrow \phi$.

- Let $S$ be a sequent $\Gamma \Rightarrow \Delta$. We define the $\mathrm{IL}$-formula $S^\sharp$ as the formula

$$\bigwedge \Gamma \to \bigvee \Delta.$$

**Theorem**

$\mathrm{IL} \vdash S^\sharp$ if and only if $\mathrm{GIL} + \mathrm{Cut} \vdash S$.

# Non-wellfounded calculus

- allows proofs of infinite height
- to guarantee that there is no vicious infinite reasoning, it is usual to add a constraint to the possible infinite paths in the proof, e.g. enforce that any infinite path goes through the premise of a rule infinite often

A **local-progress sequent calculus** is a pair $\mathcal{C} = (\mathcal{R}, L)$ where:

(i) $\mathcal{R}$ is a set of rules, called the rules of $\mathcal{C}$,

(ii) $L$ is a function that given a rule $R \in \mathcal{R}$ and an instance of the rule

$$\frac{S_0, \ldots, S_{k-1}}{S} \in R$$

returns a subset of $\{0, \ldots, k-1\}$. This subset is the set of premises of the rule instance that make progress. The function is called the local progress function of $\mathcal{C}$.

# Sequent calculus $\mathrm{G^\infty IL}$

We define the sequent calculus $\mathrm{G^\infty IL}$ as the local-progress sequent calculus given by the following rules:

$$\dfrac{}{p, \Gamma \Rightarrow p, \Delta}\,\mathsf{ax} \qquad \dfrac{}{\bot, \Gamma \Rightarrow \Delta}\,\bot\mathsf{L}$$

$$\dfrac{\Gamma \Rightarrow \Delta, \phi \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \to \psi, \Gamma \Rightarrow \Delta}\,\to\mathsf{L} \qquad \dfrac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \to \psi}\,\to\mathsf{R}$$

$$\dfrac{[\psi_i, \Phi_{[0,i)} \rhd \bot, \phi \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi]_{m \dots i \dots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta}\,\rhd_{\mathrm{IK4}}$$

Progress only occurs at the premises of $\rhd_{\mathrm{IK4}}$. The system with the Cut rule:

$$\dfrac{\Gamma \Rightarrow \Delta, \chi \qquad \chi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathsf{Cut}$$

will be denoted as $\mathrm{G^\infty IL + Cut}$.

# Comparison of rules

$$\frac{[\psi_i \rhd \bot, \psi_i, \Phi_{[0,i)} \rhd \bot, \phi \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi]_{m \dots i \dots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta} \rhd_{\mathrm{IL}}$$

$$\frac{[\psi_i, \Phi_{[0,i)} \rhd \bot, \phi \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi]_{m \dots i \dots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta} \rhd_{\mathrm{IK4}}$$

$$
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          \begin{array}{c}\vdots \\ \neg\phi, \bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \bot, \neg\psi\end{array}
          \qquad
          \dfrac{}{\bot, \ldots \Rightarrow \ldots}\,\bot\mathsf{L}
        }{\bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \phi, \bot, \Box\phi}\,\rhd_{\mathrm{IK4}}
        \qquad
        \dfrac{}{\phi, \ldots \Rightarrow \phi, \ldots}\,\mathsf{Ax}
      }{\psi, \bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \phi, \bot}\,{\to}\mathsf{L}
    }{\bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \phi, \bot, \neg\psi}\,\neg\mathsf{R}
  }{\neg\phi, \bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \bot, \neg\psi}\,\neg\mathsf{L}
  \qquad
  \dfrac{}{\bot, \ldots \Rightarrow \bot}\,\bot\mathsf{L}
}{
  \dfrac{\neg\psi \rhd \bot \Rightarrow \neg\phi \rhd \bot}{\Rightarrow \Box\psi \to \Box\phi}\,{\to}\mathsf{R}
}\,\rhd_{\mathrm{IK4}}
$$

So we have proved that $\mathrm{G}^{\infty}\mathrm{IL} \vdash \Box\psi \to \Box\phi$, where $\psi = \Box\phi \to \phi$.

# From $\mathrm{GIL} + \mathrm{Cut}$ to $\mathrm{G^{\infty}IL} + \mathrm{Cut}$

**Theorem**

Let $S$ be a sequent. If $\mathrm{GIL} + \mathrm{Cut} \vdash S$, then $\mathrm{G^{\infty}IL} + \mathrm{Cut} \vdash S$.

**Proof.**

- by induction on the height of the proof $\pi \vdash S$ in $\mathrm{GIL} + \mathrm{Cut}$ and case analysis in the last rule applied

# Local cut admissibility

If $\pi$ is a non-wellfounded proof, we denote by $\mathsf{lhg}(\pi)$ its local height.

**Lemma**

Assume we have proofs $\pi \vdash \Gamma \Rightarrow \Delta, \chi$ and $\tau \vdash \chi, \Gamma \Rightarrow \Delta$ in $\mathrm{G}^{\infty}\mathrm{IL}$ + $\mathrm{Cut}$ which are locally cut-free. Then there is $\rho \vdash \Gamma \Rightarrow \Delta$ in $\mathrm{G}^{\infty}\mathrm{IL}$ + $\mathrm{Cut}$ which is locally cut-free.

**Proof**. By induction on the lexicographic order of the pairs $\langle |\chi|, \mathsf{lhg}(\pi) + \mathsf{lhg}(\tau) \rangle$.

# Coalgebraic Proof Translations of Non-Wellfounded Proofs

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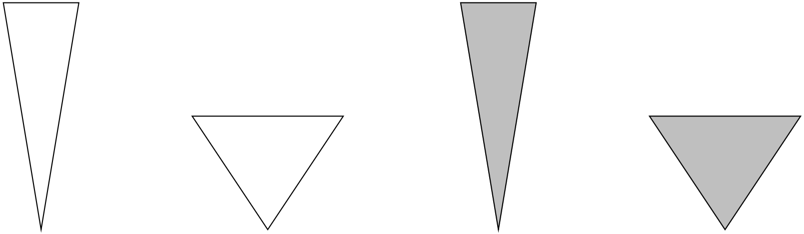
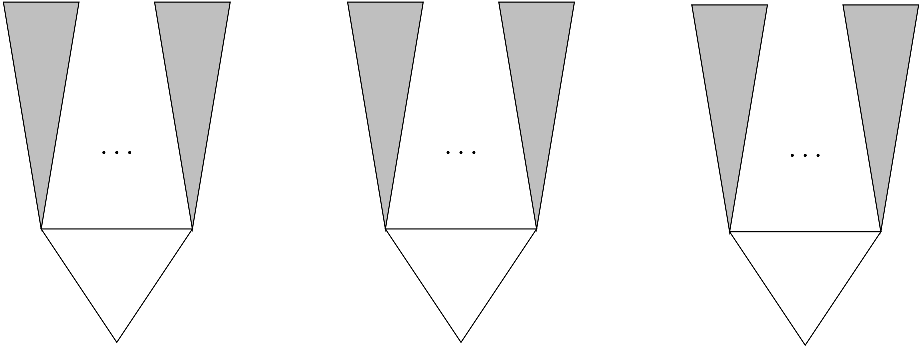


On the left: non-wellfounded proof and local fragment without cuts;
on the right: non-wellfounded proof and local fragment that can contain cuts

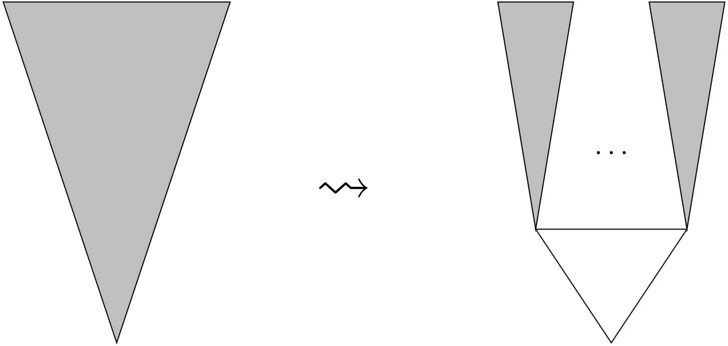
# Coalgebraic Proof Translations of Non-Wellfounded Proofs

(1) local admissibility



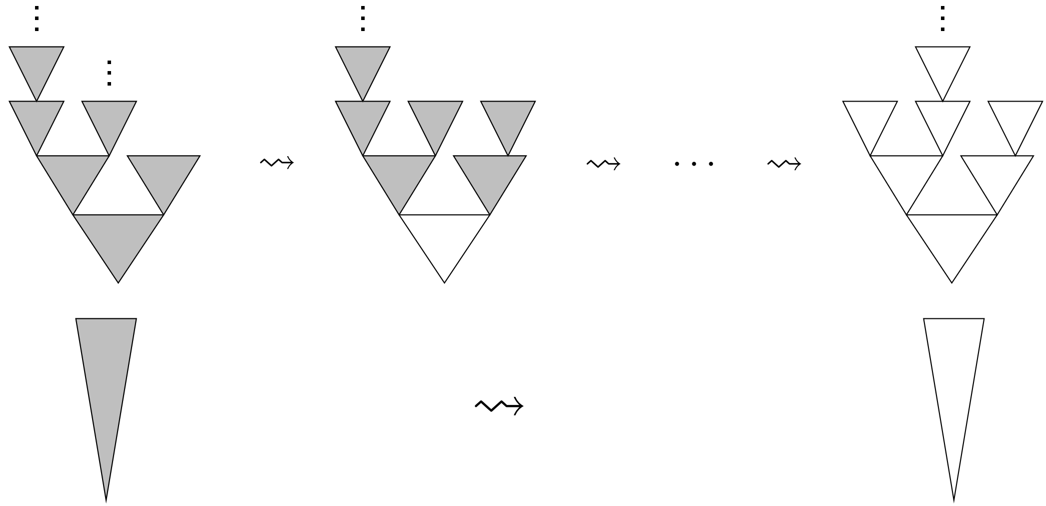
$\Gamma \Rightarrow \Delta, \chi$ $\qquad$ $\chi, \Gamma \Rightarrow \Delta$ $\qquad$ $\Gamma \Rightarrow \Delta$

(2) step (local eliminability)



$$\Gamma \Rightarrow \Delta \quad \rightsquigarrow \quad \Gamma \Rightarrow \Delta$$

(3) applying step (eliminability)



# Cut elimination for $\mathrm{G}^{\infty}\mathrm{IL}$

By just applying the previous lemma corecursively to local proof fragments we get the desired result about the cut elimination for $\mathrm{G}^{\infty}\mathrm{IL}$.

**Theorem**

If $\mathrm{G}^{\infty}\mathrm{IL} + \mathrm{Cut} \vdash S$, then $\mathrm{G}^{\infty}\mathrm{IL} \vdash S$.

# The set $Sub(\phi)$

- Let $\phi$ be a formula. We define the set $\mathsf{Sub}(\phi)$ recursively as follows:

$$\begin{aligned}
\mathsf{Sub}(p) &= \{p\},\\
\mathsf{Sub}(\bot) &= \{\bot\},\\
\mathsf{Sub}(\phi \to \psi) &= \{\phi \to \psi\} \cup \mathsf{Sub}(\phi) \cup \mathsf{Sub}(\psi),\\
\mathsf{Sub}(\phi \rhd \psi) &= \{\phi \rhd \psi, \phi \rhd \bot, \psi \rhd \bot, \bot\} \cup \mathsf{Sub}(\phi) \cup \mathsf{Sub}(\psi).
\end{aligned}$$

- If $\Gamma$ is a multiset of formulas, $\mathsf{Sub}(\Gamma)$ is the *set*

$$\bigcup \{\mathsf{Sub}(\phi) \mid \phi \in \Gamma\}.$$

- If $S = (\Gamma \Rightarrow \Delta)$ is a sequent, then $\mathsf{Sub}(S)$ is simply the set $\mathsf{Sub}(\Gamma \cup \Delta)$.

$$\frac{[\psi_i, \Phi_{[0,i)} \rhd \bot, \phi \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi]_{m \ldots i \ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta} \rhd_{\mathrm{IK4}}$$

# Lemma - formulas occuring in $\mathrm{G}^{\infty}\mathrm{IL}$-proofs

**Lemma**

Let $\pi \vdash S$ in $\mathrm{G}^{\infty}\mathrm{IL}$ and $\phi$ be a formula occurring in $\pi$. Then $\phi \in \mathsf{Sub}(S)$.

**Proof.** By the induction on the length of the node where $\phi$ appears.

# Cut elimination for $\mathrm{GIL}$

**Theorem**

For any finite set $\Lambda$ of formulas, we have that

$$\mathrm{G^{\infty}IL} \vdash \Gamma \Rightarrow \Delta \text{ implies } \mathrm{GIL} \vdash \Lambda \rhd \bot, \Gamma \Rightarrow \Delta.$$

**Proof.** Let $\pi \vdash \Gamma \Rightarrow \Delta$ in $\mathrm{G^{\infty}IL}$. By induction on the lexicographical order

$$\Big\langle |\mathsf{Sub}(\Gamma \Rightarrow \Delta) \setminus \Lambda|, \mathsf{lhg}(\pi) \Big\rangle$$

and the case analysis in the last rule of $\pi$.

**Corollary**

Let $S$ be a sequent. If $\mathrm{GIL} + \mathrm{Cut} \vdash S$, then $\mathrm{GIL} \vdash S$.

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Questions?