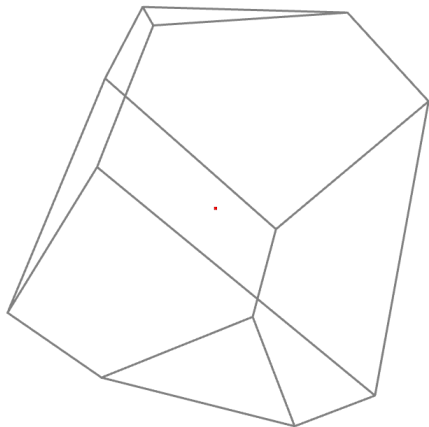


Two Extensions of Lambek Calculus

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Plan

SWOT Strengths and weaknesses of two extensions of Lambek Calculus:

- ▶ Unary modalities, structural control
 running example: relative clauses
- ▶ Grishin, polarities, continuations
 scope construal

But first a little appetizer . . .

Bluff your way in grammar logics

Grammar logics in the style of [Lam68] ‘Deductive systems as categories’. One starts from Id/Composition, plus inference rules characterizing the *logical* properties of the type-forming operations. As further add-ons, one can then add structural postulates.

$$A \longrightarrow A \qquad \frac{A \longrightarrow B \quad B \longrightarrow C}{A \longrightarrow C}$$

Logic Laws of adjointness. At the core of the grammar logics are residuated families. The base logic **NL** of [Lam61] has a multiplicative product family. [Gris83] adds a multiplicative sum family, related to the former by an arrow reversal symmetry.

$$A \longrightarrow C/B \quad \text{iff} \quad A \otimes B \longrightarrow C \quad \text{iff} \quad B \longrightarrow A \backslash C$$

$$B \odot C \longrightarrow A \quad \text{iff} \quad C \longrightarrow B \oplus A \quad \text{iff} \quad C \oslash A \longrightarrow B$$

NL_◇ [MM96] adds a pair of *unary* residuated operations.

$$\diamond A \longrightarrow B \quad \text{iff} \quad A \longrightarrow \Box B$$

Structure Same-sort associativity/commutativity.

Adding $\otimes$ associativity to **NL**, one obtains **L**, the Syntactic Calculus of [Lam58].

Adding $\otimes$ commutativity to **L**, one obtains **LP**, a.k.a. the $\otimes, \multimap$ fragment of ILL (Intuitionistic Linear Logic). Right/left division $/, \backslash$ collapse to linear implication $\multimap$. Grishin's right/left difference operations $\oslash, \oslash$ collapse to subtraction $A - B$.

$$A \otimes (B \otimes C) \multimap (A \otimes B) \otimes C \quad ; \quad A \oplus (B \oplus C) \multimap (A \oplus B) \oplus C$$

$$A \otimes B \longrightarrow B \otimes A \quad ; \quad A \oplus B \longrightarrow B \oplus A$$

NL $_{\diamond}$: controlled associativity/commutativity, keyed to $\diamond$

$$(A \otimes B) \otimes \diamond C \longrightarrow A \otimes (B \otimes \diamond C) \quad (A \otimes B) \otimes \diamond C \longrightarrow (A \otimes \diamond C) \otimes B$$

Linear distributivity laws 'mixed' associativity and/or commutativity.

One-way 'entropy laws' (direction $\leftarrow$ or $\rightarrow$) or bidirectional isomorphisms.

$$A \oslash (B \otimes C) \multimap (A \oslash B) \otimes C \quad ; \quad (A \otimes B) \oslash C \multimap A \otimes (B \oslash C)$$

$$A \oslash (B \otimes C) \multimap B \otimes (A \oslash C) \quad ; \quad (A \otimes B) \oslash C \multimap (A \oslash C) \otimes B$$

Down the rabbit hole Units, E.M. laws, compactness, ...



Unary modalities, structural control

A landscape of logics

Lambek calculi Identity $A \longrightarrow A$, composition $A \longrightarrow C$ if $A \longrightarrow B$ and $B \longrightarrow C$

Residuation: $B \longrightarrow A \backslash C$ iff $A \bullet B \longrightarrow C$ iff $A \longrightarrow C / B$

Options: $\bullet$ associativity and/or commutativity; multiplicative unit

Substructural, sublinear a hierarchy of type logics reflecting different views on the **structure** of the assumptions Γ in sequent judgements $\Gamma \vdash A$.

LOGIC	Γ	ASS	COMM
LP	multiset	✓	✓
L	string	✓	-
NL	tree	-	-

► (N)L: syntactic types

NL types assigned to phrases (bracketed strings); L: types assigned to strings

► LP (aka unit-free MILL): semantic types aka unit-free MILL

Models: residuated monoids/groupoids

(N)L intended models for the **syntactic** calculi are the multiplicative systems freely generated by the words of the language under concatenation.

Types as sets of expressions, i.e. subsets of a groupoid/semigroup/monoid $\langle M, \cdot \rangle$ with

$$\begin{aligned} A \bullet B &= \{a \cdot b \in M \mid a \in A \wedge b \in B\} \\ C/B &= \{a \in M \mid \forall_{b \in B} a \cdot b \in C\} \\ A \backslash C &= \{b \in M \mid \forall_{a \in A} a \cdot b \in C\} \\ I &= \{1\} \end{aligned}$$

- ▶ groupoid [L61], types assigned to **phrases**, bracketed strings
- ▶ semigroup [L58], types assigned to **strings**, associative multiplication
- ▶ monoid [L88], multiplicative unit, empty string

Parsing = deduction

Displaying derivations in N.D. format left of turnstile: words instead of their types

$$\begin{array}{c}
 \text{paper} \quad \text{that} \quad \text{rejected} \\
 \hline
 n \quad (n \backslash n) / (s / np) \quad np \quad (np \backslash s) / np \quad np \vdash np \\
 \hline
 \text{that} \cdot (Bob \cdot \text{rejected}) \vdash n \backslash n \quad \text{rejected} \cdot np \vdash np \backslash s \quad /E \\
 \hline
 \text{paper} \cdot (\text{that} \cdot (Bob \cdot \text{rejected})) \vdash n \quad \backslash E \\
 \hline
 \text{paper} \cdot (\text{that} \cdot (Bob \cdot \text{rejected})) \vdash n
 \end{array}$$

- ▶ axiom leaves: lexical type assignments; A^r semi-associativity
- ▶ $/, \backslash E$: slash Elim $\simeq$ modus ponens; $/, \backslash I$: slash Intro $\simeq$ hypothetical reasoning

Alternative formats sequent calculus, display logic, proof nets, ...

Grammars

A categorial grammar consists of a **universal** and a **language-specific** component.

- ▶ universal: a type calculus, e.g. **(N)L**
- ▶ language specific: a lexicon assigning each word a finite number of types

Language Given a categorial grammar G and a type B we write $L(G, B)$ for the strings of type B recognized by G . $w_1 \cdots w_n \in L(G, B)$ if the following hold:

- ▶ $(w_i, A_i) \in \text{Lex}$ for $1 \leq i \leq n$;
- ▶ $\Gamma_{[A_1, \dots, A_n]} \vdash B$, for Γ an antecedent structure with yield $A_1, \dots, A_n$

Idealization?

From syntax to semantics

The classical view compositional interpretation as homomorphism Montague 1970,
Universal Grammar

$$\text{Source} \xrightarrow{h} \text{Target}$$

relating types/proofs of a Source logic to their Target counterparts.

A chained view Interpretation as a two-step process $h'' \circ h'$

$$\text{Source} \xrightarrow{h'} \text{Target}_{\text{der}} \xrightarrow{h''} \text{Target}_{\text{lex}}$$

- ▶ h' **derivational** semantics, source constants (words) as black boxes
- ▶ h'' **lexical** semantics, unpacking word-internal semantics

N.D. Proofs and terms: syntactic calculi (N)L_{/, \}

Types, terms p atomic

$$A, B ::= p \mid A \backslash B \mid B / A \quad M, N ::= x \mid \lambda^r x. M \mid \lambda^l x. M \mid (M \ltimes N) \mid (N \rtimes M)$$

Wansing, 1990, Formulas-as-types for a Hierarchy of Sublogics of Int Prop Logic

Typing rules Axiom $x : A \vdash x : A$

var Γ, Δ all distinct

$$\frac{\Gamma \cdot x : A \vdash M : B}{\Gamma \vdash \lambda^r x. M : B / A} I/ \quad \frac{x : A \cdot \Gamma \vdash M : B}{\Gamma \vdash \lambda^l x. M : A \backslash B} I \backslash$$

$$\frac{\Gamma \vdash M : B / A \quad \Delta \vdash N : A}{\Gamma \cdot \Delta \vdash (M \ltimes N) : B} E/ \quad \frac{\Gamma \vdash N : A \quad \Delta \vdash M : A \backslash B}{\Gamma \cdot \Delta \vdash (N \rtimes M) : B} E \backslash$$

Compare: LP_—. **L** extended with product commutativity, a.k.a. MILL, Multiplicative Intuitionistic Linear Logic. In MILL, the slashes $/, \backslash$ collapse to linear implication $\multimap$.

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \multimap B} (\multimap I) \quad \frac{\Gamma \vdash M : A \multimap B \quad \Delta \vdash N : A}{\Gamma, \Delta \vdash M N : B} (\multimap E)$$

From (N)L to LP/MILL

Source atoms: s, np, n ; target atoms e (entities), t (truth values).

$$(\mathbf{N})\mathbf{L}_{/, \backslash}^{s, np, n} \xrightarrow{[\cdot]} \mathbf{LP/MILL}_{\multimap}^{e, t}$$

Types $[s] = t$, $[np] = e$, $[n] = e \multimap t$, $[A \backslash B] = [B/A] = [A] \multimap [B]$.

Proofs $[x] = \tilde{x}$ translates Axioms; for Intro/Elim rules:

$$[\lambda^l x.M] = [\lambda^r x.M] = \lambda \tilde{x}.[M] \quad [N \rtimes M] = [M \ltimes N] = [M] [N]$$

Example

$$\begin{aligned} M &= \text{paper} \rtimes (\text{that} \ltimes \lambda^r x.(\text{Bob} \rtimes (\text{rejected} \ltimes x))) : n \\ [M] &= (([\text{that}] \lambda x.([\text{rejected}] x) [\text{Bob}])) [\text{paper}] : e \multimap t \end{aligned}$$

Remark $[\cdot]$ sends source atoms to target **types**, not necessarily atomic.

The need for control

- ▶ languages exhibit phenomena that seem to require some form of
reordering, restructuring, copying
- ▶ global structural options are problematic
too little (undergeneration), too much (overgeneration)
- ▶ extended type language with modalities for structural control:
 - ▷ **licensing** structural reasoning that is lacking by default
 - ▷ **blocking** structural reasoning that would otherwise be available

Global associativity ☹️

Recall our relative clause example, derivable in **L** thanks to global associativity.

$$\begin{array}{c}
 \text{paper} \quad \frac{\text{that}}{(n \backslash n) / (s / np)} \\
 \hline
 n \\
 \hline
 \text{that} \cdot (\text{Bob} \cdot \text{rejected}) \vdash n \backslash n \\
 \hline
 \text{paper} \cdot (\text{that} \cdot (\text{Bob} \cdot \text{rejected})) \vdash n \quad \backslash E
 \end{array}
 \quad
 \begin{array}{c}
 \text{rejected} \\
 \hline
 \text{Bob} \quad \frac{(np \backslash s) / np \quad np \vdash np}{\text{rejected} \cdot np \vdash np \backslash s} / E \\
 \hline
 np \\
 \hline
 \text{Bob} \cdot (\text{rejected} \cdot np) \vdash s \\
 \hline
 (\text{Bob} \cdot \text{rejected}) \cdot np \vdash s \quad A^r \\
 \hline
 \text{that} \cdot (\text{Bob} \cdot \text{rejected}) \vdash s / np \quad / I \\
 \hline
 \text{Bob} \cdot \text{rejected} \vdash s / np \quad / E
 \end{array}$$

► **not enough** restricted to peripheral gaps, but

paper that Bob rejected __ immediately

► **too much** insensitive to island constraints

paper that (Alice reviewed a thesis) and_{(s \backslash s) / s} (B rejected __)

Modalities for structural control

- ▶ The type language is extended with a pair of unary connectives $\Diamond, \Box$ satisfying

$$\frac{\Diamond A \longrightarrow B}{A \longrightarrow \Box B}$$

- ▶ Logic: $\Diamond, \Box$ form a residuated pair. One easily shows

compositions: $\Diamond \Box A \longrightarrow A$ (interior) $A \longrightarrow \Box \Diamond A$ (closure)

monotonicity: from $A \longrightarrow B$ infer $\Diamond A \longrightarrow \Diamond B, \Box A \longrightarrow \Box B$

- ▶ Structure: **global** rules $\leadsto$ $\Diamond$ controlled **restricted** versions, e.g.

$$A_{\Diamond}^r : (A \bullet B) \bullet \Diamond C \longrightarrow A \bullet (B \bullet \Diamond C)$$

$$C_{\Diamond}^r : (A \bullet B) \bullet \Diamond C \longrightarrow (A \bullet \Diamond C) \bullet B$$

Multimodal generalization families $\{\Diamond_i, \Box_i\}_{i \in I}$ for particular structural choices

$\Diamond, \Box$ inverse duals



Relational semantics

Frames (W, R^2, R^3) . Valuation v sends types to subsets of W ,

$$\begin{aligned}v(A \bullet B) &= \{x \mid \exists yz. Rxyz \wedge y \in v(A) \wedge z \in v(B)\} \\v(C/B) &= \{y \mid \forall xz. (Rxy \wedge z \in v(B)) \Rightarrow x \in v(C)\} \\v(A \setminus C) &= \{z \mid \forall xy. (Rxy \wedge y \in v(A)) \Rightarrow x \in v(C)\} \\v(\Diamond A) &= \{x \mid \exists y. (Rxy \wedge y \in v(A))\} \\v(\Box A) &= \{y \mid \forall x. (Rxy \Rightarrow x \in v(A))\}\end{aligned}$$

Soundness/completeness Kurtonina 1995 generalizing Došen 1992 for **(N)L(P)**

Extensions of **NL**_◊ with **weak Sahlqvist** postulates are complete w.r.t. the class of 2/3-ary frames satisfying the corresponding 1st order constraint effectively computable by the Sahlqvist-van Benthem algorithm.

Weak Sahlqvist postulates $A \longrightarrow B$ such that A is built out of single-use atoms and connectives $\bullet, \Diamond$; B also is pure $\bullet, \Diamond$ frm containing at least one occurrence of $\bullet$ or $\Diamond$, with all atoms of B occurring in A .

Control operators: N.D. rules, terms

Structures Unary $\langle \rangle$ structural counterpart of $\Diamond$: $\Gamma, \Delta ::= A \mid \langle \Gamma \rangle \mid \Gamma \cdot \Delta$

$$\frac{\langle \Gamma \rangle \vdash A}{\Gamma \vdash \Box A} \Box I \qquad \frac{\Gamma \vdash \Box A}{\langle \Gamma \rangle \vdash A} \Box E$$

$$\frac{\Gamma \vdash A}{\langle \Gamma \rangle \vdash \Diamond A} \Diamond I \qquad \frac{\Delta \vdash \Diamond A \quad \Gamma[\langle A \rangle] \vdash B}{\Gamma[\Delta] \vdash B} \Diamond E \qquad \frac{\Gamma[\langle A \rangle] \vdash B}{\Gamma[\Diamond A] \vdash B} \Diamond E'$$

shorthand $(\Diamond E')$ if left premise of $(\Diamond E)$ is an axiom

Control operators: terms Terms: $M, N ::= x \mid \dots \mid \nabla M \mid \Delta M \mid \blacktriangledown M \mid \blacktriangle M$

$$\frac{\langle \Gamma \rangle \vdash M : A}{\Gamma \vdash \blacktriangle M : \Box A} \Box I \qquad \frac{\Gamma \vdash M : \Box A}{\langle \Gamma \rangle \vdash \blacktriangledown M : A} \Box E$$

$$\frac{\Gamma \vdash M : A}{\langle \Gamma \rangle \vdash \Delta M : \Diamond A} \Diamond I \qquad \frac{\Delta \vdash M : \Diamond A \quad \Gamma[\langle x : A \rangle] \vdash N : B}{\Gamma[\Delta] \vdash N[\nabla M/x] : B} \Diamond E$$

$\Diamond E$ officially: case ∇M of x in N

Controlled associativity/commutativity ☺

$\Diamond \Box np$: 'moveable' np ; key-and-lock: contract $\Diamond \Box np$ to np , once in place.

$$\begin{array}{c}
 \text{paper} \quad \frac{\text{that}}{n} \ell \quad \frac{(n \setminus n) / (s / \Diamond \Box np)}{\text{that} \cdot (\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \vdash n \setminus n} \ell \\
 \hline
 \text{paper} \cdot (\text{that} \cdot (\text{Bob} \cdot (\text{rejected} \cdot \text{immediately}))) \vdash n \quad \backslash E \\
 \\
 \text{Bob} \quad \frac{\frac{\frac{\text{rejected}}{(np \setminus s) / np} \ell \quad \frac{\frac{\Box np \vdash \Box np}{\langle \Box np \rangle \vdash np} \Box E}{\text{rejected} \cdot \langle \Box np \rangle \vdash np \setminus s} / E \quad \frac{\text{immediately}}{(np \setminus s) \setminus (np \setminus s)} \ell}{\frac{\text{rejected} \cdot \langle \Box np \rangle \vdash np \setminus s}{(\text{rejected} \cdot \langle \Box np \rangle) \cdot \text{immediately} \vdash np \setminus s} \backslash E} \ell \\
 \hline
 \text{Bob} \cdot ((\text{rejected} \cdot \langle \Box np \rangle) \cdot \text{immediately}) \vdash s \quad C_{\Diamond}^r \\
 \text{Bob} \cdot ((\text{rejected} \cdot \text{immediately}) \cdot \langle \Box np \rangle) \vdash s \quad A_{\Diamond}^r \\
 (\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \cdot \langle \Box np \rangle \vdash s \quad \Diamond E' \\
 (\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \cdot \Diamond \Box np \vdash s \quad /I \\
 \text{Bob} \cdot (\text{rejected} \cdot \text{immediately}) \vdash s / \Diamond \Box np \quad /E \\
 \hline
 \text{that} \cdot (\text{Bob} \cdot (\text{rejected} \cdot \text{immediately})) \vdash n \setminus n \quad \backslash E
 \end{array}$$

$$A_{\Diamond}^r : (A \bullet B) \bullet \Diamond C \longrightarrow A \bullet (B \bullet \Diamond C) \quad C_{\Diamond}^r : (A \bullet B) \bullet \Diamond C \longrightarrow (A \bullet \Diamond C) \bullet B$$

Proofs and terms

Adjusted lexical meaning recipe for the relative pronoun, $(n \setminus n) / (s / \Diamond \Box np)$

$$[\text{that}]^{lex} = \lambda v \lambda w \lambda z. ((w (\blacktriangledown \nabla z)) \wedge (v z))$$

► v of type $[s / \Diamond \Box np]^{lex} = \Diamond \Box e \rightarrow t$; w of type $[n]^{lex} = e \rightarrow t$

► z reusable $\Diamond \Box e$ variable distributed over the $\wedge$ conjuncts

Proof term M , derivational $[M]^{der}$ and lexical $[M]^{lex}$ translations:

$$M = \text{paper} \rtimes (\text{that} \rtimes \lambda^r x. (\text{Bob} \rtimes ((\text{rejected} \rtimes (\blacktriangledown \nabla x))) \rtimes \text{immediately})) : n$$

$$[M]^{der} = ([\text{that}] \lambda x. (([\text{immediately}] ([\text{rejected}] (\blacktriangledown \nabla x))) [\text{Bob}])) [\text{paper}] : e \multimap t$$

$$[M]^{lex} = \lambda z. ((\text{PAPER} (\blacktriangledown \nabla z)) \wedge ((\text{IMMEDIATELY} (\text{REJECTED} (\blacktriangledown \nabla z))) \text{BOB})) : \Diamond \Box e \rightarrow t$$

Blocking structural rules

Recall the **island violations** caused by (global or controlled!) associativity:

paper that (Alice reviewed a thesis) but_{(s\ s)/s} (Bob rejected __)

$$\begin{array}{c}
 \frac{\dots}{s} \quad \frac{\text{but}}{(s \setminus \Box s)/s} \quad \frac{\frac{B}{np} \quad \frac{\frac{\text{rejected}}{(np \setminus s)/np} \quad \frac{\Box np \vdash \Box np}{\langle \Box np \rangle \vdash np}}{\text{rejected} \cdot \langle \Box np \rangle \vdash np \setminus s} /E}{B \cdot (\text{rejected} \cdot \langle \Box np \rangle) \vdash s} \setminus E \\
 \frac{\dots \cdot (\text{but} \cdot (B \cdot (\text{rejected} \cdot \langle \Box np \rangle))) \vdash s \setminus \Box s}{\dots \cdot (\text{but} \cdot (B \cdot (\text{rejected} \cdot \langle \Box np \rangle))) \vdash \Box s} /E \\
 \frac{\dots \cdot (\text{but} \cdot (B \cdot (\text{rejected} \cdot \langle \Box np \rangle))) \vdash \Box s}{\langle \dots \cdot (\text{but} \cdot (B \cdot (\text{rejected} \cdot \langle \Box np \rangle))) \rangle \vdash s} \Box E \\
 \frac{\langle \dots \cdot (\text{but} \cdot (B \cdot \text{rejected})) \rangle \cdot \langle \Box np \rangle \vdash s}{\langle \dots \cdot (\text{but} \cdot (B \cdot \text{rejected})) \rangle \cdot \langle \Box np \rangle \vdash s} \text{???}
 \end{array}$$

◇ **as an obstacle** a modified type assignment imposes the desired island constraint:

► but :: (s \ \Box s)/s

Morrill 1994, “bracket” modalities

► \Box Elim seals off the conjunction as an island from which $\langle \Box np \rangle$ cannot escape

One can generalize this idea to demarcate **dependency domains** ...

Comparing RES and BANG

Correspondences Similarities more striking than differences, reading $!_i$ as $\Diamond_i \Box_i$
 Simulating $!_i$ properties as combinations of $\Diamond, \Box$ logical and structural rules, e.g.

$$\frac{\Gamma \vdash B}{! \Gamma \vdash ! B} SP \qquad \frac{\frac{\Gamma \vdash B}{\langle \Box \rangle \Gamma \vdash B} \Box L \quad \frac{\langle \Box \Gamma \rangle \vdash B}{\Box \Gamma \vdash \Box B} K}{\Box \Gamma \vdash \Box B} \Box R$$

MM 1996

Differences some features of RES not shared by BANG

- ▶ licensing and blocking uses of modalities share same logical rules
- ▶ components $\Diamond$ and $\Box$ have individual uses, cf encoding dependency relations

Resolution? Multitype approach, Palmigiano c.s., arguing that $!$ cannot be seen as primitive, but must be deconstructed in heterogeneous adjoint pair $\Diamond \blacksquare$

From postulates to structural rules

Linearity general form of **linear** structural rules:

Moot 2002

$$\frac{\Gamma[\Xi[\Delta_1, \dots, \Delta_n]] \vdash A}{\Gamma[\Xi'[\Delta_{\pi_1}, \dots, \Delta_{\pi_n}]] \vdash A} R$$

- ▶ $\Xi[], \Xi'[]$ generalized contexts of arity n : $\mathcal{C} ::= [] \mid \langle \mathcal{C} \rangle \mid \mathcal{C} \cdot \mathcal{C}$ arity: # holes
- ▶ $\Xi[\Gamma_1, \dots, \Gamma_n]$ structure obtained by substitution of $\Gamma_1, \dots, \Gamma_n$ in $\Xi[]$ of arity n

Linear, non-increasing R is non-increasing if $|\Xi'[]| \leq |\Xi[]|$

- ▶ number of unary $\langle \rangle$ in conclusion $\leq$ in number of $\langle \rangle$ premise
- ▶ compare: $\diamond(A \bullet B) \longrightarrow \diamond A \bullet \diamond B$ ✓; but not $\diamond A \bullet \diamond B \longrightarrow \diamond(A \bullet B)$

Complexity, expressivity (Moot 2002) $\mathbf{NL}_\diamond$ + linear, non-increasing structural rules:

- ▶ decidable; PSPACE complete; recognizes the context-sensitive languages
- ▶ Mildly CS fragments? Moot 2008, simulating TAGs

Discussion

What about the thesis

grammar = universal type logic + language specific lexicon

Atoms of variation the controlled ass/comm postulates have a certain simplicity

- ▶ local patterns, global effect through recombinant qualities
- ▶ correlation with language typology:

English VO relpro $(n \setminus n) / (s / \Diamond \Box np)$ vs Dutch OV $(n \setminus n) / (\Diamond \Box np \setminus s)$

- ▶ for Serbo-Croatian cf Vermaat, 2006, The logic of variation [PhD thesis](#)

Challenges Solutions for more dramatic syn/sem mismatches tend to be laborious

- ▶ patterns beyond CF, e.g. $a^n b^n c^n$, compare naturality of k -MCFG analysis
- ▶ scope construal of generalized quantifier expressions

A radical alternative: going neural

PhD project Kogkalidis meaning composition is directly computed from surface string, forgoing explicit structural rules

- ▶ Kogkalidis, 2023, Dependency as Modality, Parsing as Permutation.
Phd Thesis, Utrecht University, Beth Dissertation Award 2024 [url](#)
- ▶ Kogkalidis & MM, 2022, [arXiv](#)
Geometry-Aware Supertagging with Heterogeneous Dynamic Convolutions
- ▶ Kogkalidis, MM & Moot, 2020
Neural Proof Nets. CoNLL [url](#)

Code: <https://github.com/konstantinosKokos/spindle>

Online demo: <https://parseport.hum.uu.nl/spindle>

Challenges

Recall we write $L(G, B)$ for the strings of type B recognized by grammar G .

$w_1 \cdots w_n \in L(G, B)$ if the following hold:

- $(w_i, A_i) \in \text{Lex}$ for $1 \leq i \leq n$;
- $\Gamma_{[A_1, \dots, A_n]} \vdash B$, for Γ an antecedent structure with yield $A_1, \dots, A_n$

► type ambiguity: what is the right type for w_i given its context?

$\leadsto$ supertagging

► structural ambiguity: what is the proper structure for Γ to derive B

$\leadsto$ parsing: neural proof nets

Integrating supertagging and neural parsing

Neural proof nets The parsing method uses LL proof nets. Proof net construction can be seen as a staged process:

- ▶ proof frame: forest of formula decomposition trees — supertagging ☺
- ▶ proof structure: p frame plus pairwise linking of in/out atoms
- ▶ proof net: p structure with successful traversal

MILL_—^{◇,□} lambda term as byproduct of traversal

Key neural methods

- ▶ supertagging: **parallel** tree decoding with dynamic graph convolutions
- ▶ axiom linking: Sinkhorn iterative method to approach double stochastic matrix
- ▶ verification: Lamarche traversal method

Lamarche 2008

Grishin, polarities, continuations

Some background refs in addition to the abstract bibliography

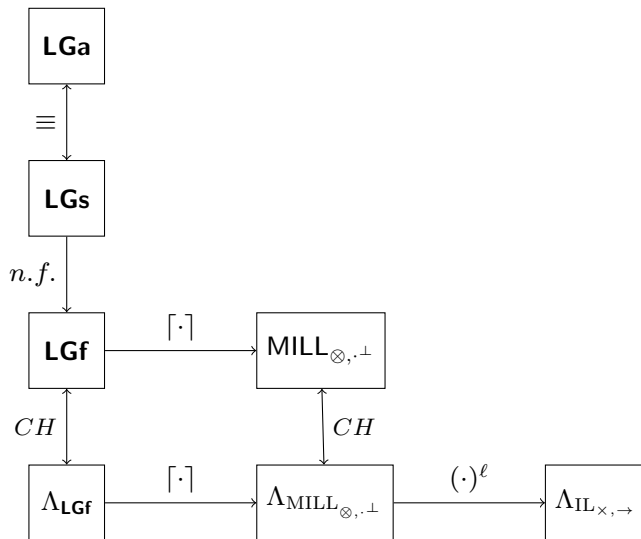
- ▶ Curien & Herbelin, The duality of computation, ICFP 2000
- ▶ Bernardi & MM, ESSLLI 2007 Course Notes:

<http://symcg.pbworks.com/f/essllinotesnew.pdf>

- ▶ Bastenhof, Polarized Montagovian Semantics for the Lambek-Grishin Calculus, FG 2010; Categorical Symmetry, PhD thesis 2013

Road map

On the left the **LG** source calculus in its various presentations; moving to the right the target calculi obtained by compositional translations (homomorphisms).



Glossary

- **LGa**: categorical presentation with arrows $f : A \longrightarrow B$. Atomic types $\mathcal{A} = \{np, n, s\}$; connectives $\{\otimes, /, \backslash, \oplus, \odot, \oslash\}$.
- **LGs**: display sequent calculus. Equivalence $\mathbf{LGa} \equiv \mathbf{LGs}$. **LGs** enjoys cut elimination. Corollary: decidability, subformula property.
- **LGf**: focalized **LG**. Elimination of spurious ambiguity: normal form derivations through polarity restrictions.

$\Lambda_{\mathbf{LGf}}$: term language, Curry-Howard isomorphic to **LGf** proofs.

- $\mathbf{MILL}_{\otimes, \perp}$: the target logic for derivational semantics. The set of atomic types is $\mathcal{A} \cup \{\perp\}$; connectives: multiplicative conjunction $\otimes$, restricted linear implication $\cdot \multimap \perp$ w.r.t. response type $\perp$. $\lceil \cdot \rceil$ translates normal **LGf** sequent proofs to normal **MILL** natural deduction proofs.

$\Lambda_{\mathbf{MILL}_{\otimes, \perp}}$: linear lambda calculus in Curry-Howard correspondence with $\mathbf{MILL}_{\otimes, \perp}$.

- $\Lambda_{\mathbf{IL}_{\times, \rightarrow}}$: target term language for lexical semantics. Substitution of (possibly non-linear) translations for lexical constants in the linear derivational **MILL** terms.

Lambek-Grishin calculus: categorical presentation

$$A, B ::= p \mid \\ A \otimes B \mid B \backslash A \mid A / B \mid \\ A \oplus B \mid A \odot B \mid B \oslash A$$

atoms: $s, np, \dots$
 product, left vs right division
 sum, right vs left difference

Identity, composition of arrows

Residuation laws

$$\begin{array}{c} \frac{f : A \otimes B \longrightarrow C}{\triangleright f : A \longrightarrow C/B} \\[1em] \frac{g : A \longrightarrow C/B}{\triangleright^{-1} g : A \otimes B \longrightarrow C} \end{array} \qquad \begin{array}{c} \frac{f : A \otimes B \longrightarrow C}{\triangleleft f : B \longrightarrow A \backslash C} \\[1em] \frac{g : B \longrightarrow A \backslash C}{\triangleleft^{-1} g : A \otimes B \longrightarrow C} \end{array}$$

Dual residuation laws

$$\begin{array}{c} \frac{f : C \longrightarrow B \oplus A}{\blacktriangleleft f : B \odot C \longrightarrow A} \\[1em] \frac{g : B \odot C \longrightarrow A}{\blacktriangleleft^{-1} g : C \longrightarrow B \oplus A} \end{array} \qquad \begin{array}{c} \frac{f : C \longrightarrow B \oplus A}{\blacktriangleright f : C \odot A \longrightarrow B} \\[1em] \frac{g : C \odot A \longrightarrow B}{\blacktriangleright^{-1} g : C \longrightarrow B \oplus A} \end{array}$$

Monotonicity laws

$$\frac{f : A \longrightarrow A' \quad g : B \longrightarrow B'}{f/g : A/B' \longrightarrow A'/B} \quad \frac{f : A \longrightarrow A' \quad g : B \longrightarrow B'}{f \otimes g : A \otimes B \longrightarrow A' \otimes B'} \quad \frac{f : A \longrightarrow A' \quad g : B \longrightarrow B'}{g \backslash f : B' \backslash A \longrightarrow B \backslash A'}$$

Dual monotonicity laws

$$\frac{f : A \longrightarrow A' \quad g : B \longrightarrow B'}{f \oslash g : A \oslash B' \longrightarrow A' \oslash B} \quad \frac{f : A \longrightarrow A' \quad g : B \longrightarrow B'}{f \oplus g : A \oplus B \longrightarrow A' \oplus B'} \quad \frac{f : A \longrightarrow A' \quad g : B \longrightarrow B'}{g \oslash f : B' \oslash A \longrightarrow B \oslash A'}$$

Linear distributivity laws (type IV, type I: invert the arrow)

$$\begin{array}{ll} \mathbf{d} : (A \oslash B) \otimes C \longrightarrow A \oslash (B \otimes C) & \mathbf{b} : C \otimes (B \oslash A) \longrightarrow (C \otimes B) \oslash A \\ \mathbf{q} : C \otimes (A \oslash B) \longrightarrow A \oslash (C \otimes B) & \mathbf{p} : (B \oslash A) \otimes C \longrightarrow (B \otimes C) \oslash A \end{array}$$

Linear distributivity laws (rule form)

$$\begin{array}{ll} \frac{f : A \otimes B \longrightarrow C \oplus D}{\mathbf{d} f : C \oslash A \longrightarrow D/B} & \frac{f : A \otimes B \longrightarrow C \oplus D}{\mathbf{b} f : B \oslash D \longrightarrow A \backslash C} \\ \frac{f : A \otimes B \longrightarrow C \oplus D}{\mathbf{q} f : C \oslash B \longrightarrow A \backslash D} & \frac{f : A \otimes B \longrightarrow C \oplus D}{\mathbf{p} f : A \oslash D \longrightarrow C/B} \end{array}$$

Exercise

For each distributivity law, Grishin gives six interderivable forms.

$$\begin{array}{ll} (a) & (b \setminus c) \otimes a \longrightarrow b \setminus (c \otimes a) \\ (b) & b \setminus (c \oplus a) \longrightarrow (b \setminus c) \oplus a \\ (c) & a \otimes (c \otimes b) \longrightarrow (a \otimes c) \otimes b \\ (d) & (a \setminus c) \otimes b \longrightarrow c \otimes (a \otimes b) \\ (e) & (a \oplus b) / c \longrightarrow a / (c \otimes b) \\ (f) & a \otimes (b \otimes c) \longrightarrow (c / a) \setminus b \end{array}$$

Prove (a)–(f), and give their combinator name.

Kripke relational interpretation MM & Kurtonina 2010

$$\begin{array}{ll} v(A \otimes B) &= \{x \mid \exists yz. R_{\otimes}xyz \wedge y \in v(A) \wedge z \in v(B)\} \\ v(C/B) &= \{y \mid \forall xz. (R_{\otimes}xyz \wedge z \in v(B)) \Rightarrow x \in v(C)\} \\ v(A \setminus C) &= \{z \mid \forall xy. (R_{\otimes}xyz \wedge y \in v(A)) \Rightarrow x \in v(C)\} \\ v(A \oplus B) &= \{x \mid \forall yz. R_{\oplus}xyz \Rightarrow (y \in v(A) \vee z \in v(B))\} \\ v(C \otimes B) &= \{y \mid \exists xz. R_{\oplus}xyz \wedge z \notin v(B) \wedge x \in v(C)\} \\ v(A \oslash C) &= \{z \mid \exists xy. R_{\oplus}xyz \wedge y \notin v(A) \wedge x \in v(C)\} \end{array}$$

LGs: display sequent calculus

Display sequent calculus:

- ▶ connective left and a right intro rule, cf regular Gentzen sequent calculus
- ▶ structural proxy for each of the connectives, not just for $\otimes$ and $\oplus$.

We use the same symbol for the logical connective and its structural counterpart, using centerdots to set off the structural version.

- ▶ **LGs** enjoys cut elimination, but suffers from spurious ambiguity

Coming soon: a focalized version; source calculus for compositional, continuation-based interpretation

Sequents of the form $\mathcal{I} \vdash \mathcal{O}$, where $\mathcal{I}$ is an input structure, $\mathcal{O}$ an output structure. Structures are built out of formulas $\mathcal{F}$ by means of the grammar below.

$$\begin{aligned}\mathcal{I} &::= \mathcal{F} \mid \mathcal{I} \cdot \otimes \cdot \mathcal{I} \mid \mathcal{I} \cdot \otimes \cdot \mathcal{O} \mid \mathcal{O} \cdot \otimes \cdot \mathcal{I} \\ \mathcal{O} &::= \mathcal{F} \mid \mathcal{O} \cdot \oplus \cdot \mathcal{O} \mid \mathcal{I} \cdot \backslash \cdot \mathcal{O} \mid \mathcal{O} \cdot / \cdot \mathcal{I}\end{aligned}\tag{1}$$

In the rules below, $A, B, \dots$ are formula variables, $X, Y, \dots$ structure variables.

Structural rules The (dual) residuation laws of the categorical presentation here take the form of structural rules. They guarantee that any formula component of a structure can be displayed on its own, left or right of the turnstile. Grishin's distributivity principles, similarly, are expressed at the structural level.

$$\frac{}{A \vdash A} \text{Ax} \quad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \text{Cut}$$

$$\frac{\frac{X \vdash Z \cdot / \cdot Y}{X \cdot \otimes \cdot Y \vdash Z} rp}{Y \vdash X \cdot \backslash \cdot Z} rp \quad \frac{\frac{Y \cdot \odot \cdot Z \vdash X}{Z \vdash Y \cdot \oplus \cdot X} drp}{Z \cdot \odot \cdot X \vdash Y} drp$$

$$\frac{X \cdot \otimes \cdot Y \vdash Z \cdot \oplus \cdot W}{Z \cdot \odot \cdot X \vdash W \cdot / \cdot Y} G1$$

$$\frac{X \cdot \otimes \cdot Y \vdash Z \cdot \oplus \cdot W}{Y \cdot \odot \cdot W \vdash X \cdot \backslash \cdot Z} G3$$

$$\frac{X \cdot \otimes \cdot Y \vdash Z \cdot \oplus \cdot W}{Z \cdot \odot \cdot Y \vdash X \cdot \backslash \cdot W} G2$$

$$\frac{X \cdot \otimes \cdot Y \vdash Z \cdot \oplus \cdot W}{X \cdot \odot \cdot W \vdash Z \cdot / \cdot Y} G4$$

Logical rules Each logical connective has a rewrite rule which replaces the logical operation by its structural proxy. In the presence of Cut, the rewrite rules are invertible.

$$\frac{A \cdot \$ \cdot B \vdash Y}{A \$ B \vdash Y} \$L \quad \$ \in \{\otimes, \odot, \circ\} \quad \frac{X \vdash A \cdot \# \cdot B}{X \vdash A \# B} \#R \quad \# \in \{\oplus, /, \backslash\}$$

In the remaining rules, $(\$R)$ for $\$ \in \{\otimes, \odot, \circ\}$ and $(\#L)$ for $\# \in \{\oplus, /, \backslash\}$, we recognize the monotonicity principles.

$$\begin{array}{cc} \frac{X \vdash A \quad Y \vdash B}{X \cdot \otimes \cdot Y \vdash A \otimes B} \otimes R & \frac{A \vdash X \quad B \vdash Y}{A \oplus B \vdash X \cdot \oplus \cdot Y} \oplus L \\[10pt] \frac{X \vdash A \quad B \vdash Y}{A \backslash B \vdash X \cdot \backslash \cdot Y} \backslash L & \frac{X \vdash A \quad B \vdash Y}{X \cdot \odot \cdot Y \vdash A \odot B} \odot R \\[10pt] \frac{X \vdash A \quad B \vdash Y}{B/A \vdash Y \cdot / \cdot X} /L & \frac{X \vdash A \quad B \vdash Y}{Y \cdot \circ \cdot X \vdash B \circ A} \circ R \end{array}$$

Cut elimination The rules satisfy Belnap's (*JPL*, 1982) conditions for cut-elimination.

LGf: focused sequent calculus

Three types of sequent: neutral $X \vdash Y$, right focused $X \vdash \boxed{A}$ and left focused $\boxed{A} \vdash Y$.

- ▶ neutral sequents: display rules, distributivity rules and invertible rewrite rules operate on neutral sequents.
- ▶ applicability of the remaining rules is constrained by the *polarity* of the formulas in focus.
 - ▷ formulas with invertible left introduction rule are *positive* ($\otimes, \odot, \oslash$);
 - ▷ formulas with invertible right introduction rule are *negative* ($\oplus, /, \backslash$).
 - ▷ atomic formulas are assigned an arbitrary polarity *bias*.

Axiom, Co-axiom In the (Ax) case, A has to be positive; in the (CoAx) case A is negative.

$$\frac{}{A \vdash \boxed{A}} \text{Ax} \qquad \frac{}{\boxed{A} \vdash A} \text{CoAx}$$

Focusing, Defocusing The rules in the left column have A negative; for those in the right column, A has to be positive.

$$\frac{\boxed{A} \vdash Y}{A \vdash Y} \leftarrow \qquad \frac{X \vdash \boxed{A}}{X \vdash A} \rightarrow$$

$$\frac{X \vdash A}{X \vdash \boxed{A}} \rightarrow \qquad \frac{A \vdash Y}{\boxed{A} \vdash Y} \leftarrow$$

Monotonicity rules The non-invertible rules transfer the focus from a complex formula to its subformulae.

$$\begin{array}{c}
 \frac{X \vdash \boxed{A} \quad Y \vdash \boxed{B}}{X \cdot \otimes \cdot Y \vdash \boxed{A \otimes B}} \otimes R \qquad \frac{\boxed{A} \vdash X \quad \boxed{B} \vdash Y}{\boxed{A \oplus B} \vdash X \cdot \oplus \cdot Y} \oplus L \\
 \\
 \frac{X \vdash \boxed{A} \quad \boxed{B} \vdash Y}{\boxed{A \setminus B} \vdash X \cdot \setminus \cdot Y} \setminus L \qquad \frac{X \vdash \boxed{A} \quad \boxed{B} \vdash Y}{X \cdot \oslash \cdot Y \vdash \boxed{A \oslash B}} \oslash R \\
 \\
 \frac{X \vdash \boxed{A} \quad \boxed{B} \vdash Y}{\boxed{B / A} \vdash Y \cdot / \cdot X} /L \qquad \frac{X \vdash \boxed{A} \quad \boxed{B} \vdash Y}{Y \cdot \odot \cdot X \vdash \boxed{B \odot A}} \odot R
 \end{array}$$

Exercise Below a derivation for ‘Alice thinks everybody left’, where ‘everybody’ is typed $s/(np \backslash s)$. With the atom bias discussed in class (s negative, other atoms positive), there is only one derivation obeying the polarity restrictions. From the backward chaining perspective (conclusion to axioms), this derivation first activates the type for ‘thinks’ $(np \backslash s)/s$; activation of the type for ‘everybody’ $s/(np \backslash s)$ is delayed to the point where the embedded sentence ‘everybody left’ is derived.

Try the opposite order of activation ($s/(np \backslash s)$ before $(np \backslash s)/s$), and show where/why the derivation crashes.

(You can silently perform a sequence of residuation inferences, started and ended with (de)focusing, for example $\Leftarrow$ below.)

$$\frac{\boxed{A} \vdash Y}{X \vdash \boxed{B}} \Leftarrow \quad \text{abbreviates} \quad \frac{\boxed{A} \vdash Y}{(residuation\ steps)} \Leftarrow \quad \frac{}{X \vdash \boxed{B}} \rightarrow$$

$$\begin{array}{c}
\frac{\overline{np \vdash \boxed{np}} \quad \overline{\boxed{s} \vdash s}}{\overline{\boxed{np \setminus s} \vdash np \cdot \setminus \cdot s}} (\setminus L) \\
\frac{\overline{\boxed{np \setminus s} \vdash np \cdot \setminus \cdot s}}{np \setminus s \vdash np \cdot \setminus \cdot s} \swarrow \\
\frac{np \setminus s \vdash np \cdot \setminus \cdot s}{np \setminus s \vdash np \setminus s} (\setminus R) \\
\frac{\overline{\boxed{s} \vdash s} \quad np \setminus s \vdash \boxed{np \setminus s}}{np \setminus s \vdash s \cdot / \cdot (np \setminus s)} \neg \\
\frac{\overline{\boxed{s} \vdash s} \quad np \setminus s \vdash \boxed{np \setminus s}}{\overline{\boxed{s/(np \setminus s)} \vdash s \cdot / \cdot (np \setminus s)}} (/L) \\
\frac{\overline{\boxed{s/(np \setminus s)} \vdash s \cdot / \cdot (np \setminus s)}}{s/(np \setminus s) \vdash s \cdot / \cdot (np \setminus s)} \swarrow \\
\frac{s/(np \setminus s) \vdash s \cdot / \cdot (np \setminus s)}{(s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s) \vdash s} \neg \\
\frac{(s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s) \vdash s}{(s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s) \vdash \boxed{s}} (/L) \\
\frac{\overline{\boxed{np \setminus s}/s} \vdash (np \cdot \setminus \cdot s) \cdot / \cdot ((s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s))}{\overline{\boxed{(np \setminus s)/s} \vdash (np \cdot \setminus \cdot s) \cdot / \cdot ((s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s))}} \swarrow \\
\frac{\overline{\boxed{(np \setminus s)/s} \vdash (np \cdot \setminus \cdot s) \cdot / \cdot ((s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s))}}{((np \setminus s)/s) \cdot \otimes \cdot ((s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s)) \vdash np \cdot \setminus \cdot s} \neg \\
\frac{((np \setminus s)/s) \cdot \otimes \cdot ((s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s)) \vdash np \cdot \setminus \cdot s}{np \cdot \otimes \cdot (((np \setminus s)/s) \cdot \otimes \cdot ((s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s))) \vdash s} \neg \\
\frac{np \cdot \otimes \cdot (((np \setminus s)/s) \cdot \otimes \cdot ((s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s))) \vdash s}{np \cdot \otimes \cdot (((np \setminus s)/s) \cdot \otimes \cdot ((s/(np \setminus s)) \cdot \otimes \cdot (np \setminus s))) \vdash \boxed{s}} \neg
\end{array}$$

Solution The attempt below displays the $s/(np \backslash s)$ formula, brings it in focus and tries to decompose it by means of $(/L)$. In order to proceed, the left premise with focused $\boxed{s}$ would have to be turned into a neutral sequent by $(\neg)$. But this rule requires a *positive* formula, and s has negative bias.

$$\begin{array}{c}
 \overline{\boxed{s} \vdash ((np \backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)} \quad \overline{np \backslash s \vdash \boxed{np \backslash s}} \quad \neg \\
 \hline
 \boxed{s/(np \backslash s)} \vdash (((np \backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np \backslash s) \quad (/L) \\
 \hline
 s/(np \backslash s) \vdash (((np \backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np \backslash s) \quad \neg \\
 \hline
 (s/(np \backslash s)) \cdot \otimes \cdot (np \backslash s) \vdash ((np \backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s) \\
 \hline
 ((np \backslash s)/s) \cdot \otimes \cdot ((s/(np \backslash s)) \cdot \otimes \cdot (np \backslash s)) \vdash np \cdot \backslash \cdot s \\
 \hline
 np \cdot \otimes \cdot (((np \backslash s)/s) \cdot \otimes \cdot ((s/(np \backslash s)) \cdot \otimes \cdot (np \backslash s))) \vdash s
 \end{array}$$

LGf: proofs and terms

Λ_{LGf} : term language in Curry-Howard (i.e. 1-to-1) correspondence with **LGf** proofs.

Λ_{LGf} has three types of terms, corresponding to the three types of sequents: *values* for sequents with focus right; evaluation *contexts* for sequents with focus left; *commands* for neutral sequents.

The grammar of raw terms is given below.

$$v ::= \mu\alpha.C \mid V \quad ; \quad V ::= x \mid v_1 \otimes v_2 \mid v \odot e \mid e \odot v$$

$$e ::= \tilde{\mu}x.C \mid E \quad ; \quad E ::= \alpha \mid e_1 \oplus e_2 \mid v \backslash e \mid e / v$$

$$c ::= \langle x \upharpoonright E \rangle \mid \langle V \upharpoonright \alpha \rangle$$

$$C ::= c \mid \frac{x \ y}{z}.C \mid \frac{x \ \beta}{z}.C \mid \frac{\beta \ x}{z}.C \mid \frac{\alpha \ \beta}{\gamma}.C \mid \frac{x \ \beta}{\gamma}.C \mid \frac{\beta \ x}{\gamma}.C$$

Typing rules LGf inference rules as typing rules for the raw term language.

$$\frac{}{x : A \vdash \boxed{x : A}} \text{Ax} \qquad \frac{}{\boxed{\alpha : A} \vdash \alpha : A} \text{CoAx}$$

In the $(\rightarrow)$, $(\leftarrow)$ rules, $\mu, \tilde{\mu}$ are binding the (co)variables α, x .

$$\frac{\boxed{E : A} \vdash Y}{x : A \vdash Y} \leftarrow_{\langle x | E \rangle} \qquad \frac{X \vdash \boxed{V : A}}{X \vdash \alpha : A} \rightarrow_{\langle V | \alpha \rangle}$$

$$\frac{X \vdash \overset{C}{\alpha} : A}{X \vdash \boxed{\mu \alpha. C : A}} \rightarrow \qquad \frac{x : A \vdash \overset{C}{Y}}{\boxed{\tilde{\mu} x. C : A} \vdash Y} \leftarrow$$

Typing rules (cont'd) Rewrite rules take a structure consisting of two labelled formulas in the premise and replace it by a labelled formula in the conclusion. Below the rules for $(\otimes L)$ and $(\oslash L)$. For the other rewrite rules, apply the $(\cdot)^\natural$ and $(\cdot)^\dagger$ symmetries.

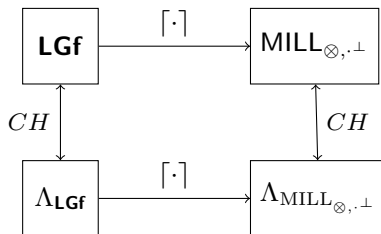
$$\frac{x : A \cdot \otimes \cdot y : B \overset{C}{\vdash} X}{z : A \otimes B \overset{\frac{x \cdot y}{z}, C}{\vdash} X} \otimes L \qquad \frac{x : A \cdot \oslash \cdot \beta : B \overset{C}{\vdash} X}{z : A \oslash B \overset{\frac{x \cdot \beta}{z}, C}{\vdash} X} \oslash L$$

Non-invertible monotonicity rules simply use the connective symbols in the term language to record the logical rule applied. Add the $(\cdot)^\natural$ and $(\cdot)^\dagger$ symmetric cases.

$$\frac{\boxed{e_1 : B} \vdash Y \quad \boxed{e_2 : A} \vdash X}{\boxed{e_1 \oplus e_2 : B \oplus A} \vdash Y \cdot \oplus \cdot X} \oplus L \qquad \frac{\boxed{e : B} \vdash Y \quad X \vdash \boxed{v : A}}{\boxed{e/v : B/A} \vdash Y \cdot / \cdot X} /L$$

The structural rules (display rules, distributivity rules) leave no trace in the term as they do not affect the interpretation.

Compositional interpretation: from **LGf** to $\text{MILL}_{\otimes, \perp}$



The map $[\cdot]$ sends **LGf** sequent derivations/terms to Natural Deduction proofs/terms in $\text{MILL}_{\otimes, \perp}$.

- ▶ $\text{MILL}_{\otimes, \perp}$: the fragment of Multiplicative Intuitionistic Logic with linear products and restricted implications $A \multimap \perp$
- ▶ atom $\perp$: response type; $A \multimap \perp$ abbreviates as $A^\perp$.

Interpretation: types By taking into account the polarities, the interpretation of types avoids the ‘bureaucratic redexes’ we saw in the Plotkin/Barker translations. For positive atoms $\lceil p \rceil = p$; for atoms with negative bias $\lceil p \rceil = p^\perp$. Complex types:

positive	A	B	$\lceil A \otimes B \rceil$	$\lceil A \oslash B \rceil$	$\lceil B \odot A \rceil$
	$-$	$-$	$\lceil A \rceil^\perp \otimes \lceil B \rceil^\perp$	$\lceil A \rceil^\perp \otimes \lceil B \rceil$	$\lceil B \rceil \otimes \lceil A \rceil^\perp$
	$-$	$+$	$\lceil A \rceil^\perp \otimes \lceil B \rceil$	$\lceil A \rceil^\perp \otimes \lceil B \rceil^\perp$	$\lceil B \rceil^\perp \otimes \lceil A \rceil^\perp$
	$+$	$-$	$\lceil A \rceil \otimes \lceil B \rceil^\perp$	$\lceil A \rceil \otimes \lceil B \rceil$	$\lceil B \rceil \otimes \lceil A \rceil$
	$+$	$+$	$\lceil A \rceil \otimes \lceil B \rceil$	$\lceil A \rceil \otimes \lceil B \rceil^\perp$	$\lceil B \rceil^\perp \otimes \lceil A \rceil$
negative	A	B	$\lceil A/B \rceil$	$\lceil B \backslash A \rceil$	$\lceil A \oplus B \rceil$
	$-$	$-$	$\lceil A \rceil \otimes \lceil B \rceil^\perp$	$\lceil B \rceil^\perp \otimes \lceil A \rceil$	$\lceil A \rceil \otimes \lceil B \rceil$
	$-$	$+$	$\lceil A \rceil \otimes \lceil B \rceil$	$\lceil B \rceil \otimes \lceil A \rceil$	$\lceil A \rceil \otimes \lceil B \rceil^\perp$
	$+$	$-$	$\lceil A \rceil^\perp \otimes \lceil B \rceil^\perp$	$\lceil B \rceil^\perp \otimes \lceil A \rceil^\perp$	$\lceil A \rceil^\perp \otimes \lceil B \rceil$
	$+$	$+$	$\lceil A \rceil^\perp \otimes \lceil B \rceil$	$\lceil B \rceil \otimes \lceil A \rceil^\perp$	$\lceil A \rceil^\perp \otimes \lceil B \rceil^\perp$

Interpretation: proofs/terms The monotonicity rules uniformly translate as linear pairs, with the translation of the rewrite rules as the matching case construct. Application/abstraction at the MILL target end arise as the image of the (de)focusing rules.

$$\begin{array}{ll}
\text{(co)var:} & \llbracket x \rrbracket = \tilde{x} \quad ; \quad \llbracket \alpha \rrbracket = \tilde{\alpha} \\
\text{linear application:} & \llbracket \langle x \upharpoonright E \rangle \rrbracket = (\tilde{x} \llbracket E \rrbracket) \quad ; \quad \llbracket \langle V \upharpoonright \alpha \rangle \rrbracket = (\tilde{\alpha} \llbracket V \rrbracket) \\
\text{linear abstraction:} & \llbracket \tilde{\mu}x.C \rrbracket = \lambda \tilde{x}. \llbracket C \rrbracket \quad ; \quad \llbracket \mu\alpha.C \rrbracket = \lambda \tilde{\alpha}. \llbracket C \rrbracket \\
\text{linear pair:} & \llbracket \phi \# \psi \rrbracket = \langle \llbracket \phi \rrbracket, \llbracket \psi \rrbracket \rangle \quad (\# \in \{\otimes, /, \backslash, \oplus, \odot, \oslash\}) \\
\text{case:} & \llbracket \frac{\phi}{\xi}.C \rrbracket = \text{case } \tilde{\xi} \text{ of } \langle \tilde{\phi}, \tilde{\psi} \rangle. \llbracket C \rrbracket
\end{array}$$

In order to extend $\llbracket \cdot \rrbracket$ to sequents, add negation for the type of negative hypotheses and positive conclusions.

$$\begin{array}{c|cc}
A & \llbracket x : A \rrbracket & \llbracket \alpha : A \rrbracket \\
\hline
+ & \tilde{x} : \llbracket A \rrbracket & \tilde{\alpha} : \llbracket A \rrbracket^\perp \\
- & \tilde{x} : \llbracket A \rrbracket^\perp & \tilde{\alpha} : \llbracket A \rrbracket
\end{array}$$

Effect of the translation Normal (i.e. cut free) **LGf** sequent proofs are mapped to normal (i.e. redex free) $\text{MILL}_{\otimes, \perp}$ N.D. proofs:

source: LGf sequent proofs	$\leadsto$	target: $\text{MILL}_{\otimes, \perp}$ N.D. proofs
$X \overset{C}{\vdash} Y$	$\leadsto$	$[X], [Y] \vdash [C] : \perp$
$X \vdash \boxed{v : A}$	$\leadsto$	$[X] \vdash [v] : [A] \quad (A : +)$
	$\leadsto$	$[X] \vdash [v] : [A]^\perp \quad (A : -)$
$\boxed{e : A} \vdash Y$	$\leadsto$	$[Y] \vdash [e] : [A] \quad (A : -)$
	$\leadsto$	$[Y] \vdash [e] : [A]^\perp \quad (A : +)$

Example We compute the interpretation of $(\dagger)$ (negative bias for s , positive for np, n).

$$(\dagger) \quad ((\text{some} : (np/n) \cdot \otimes \cdot \text{teacher} : n) \cdot \otimes \cdot \text{left} : (np \setminus s)) \vdash \boxed{s}$$

$$\begin{array}{c}
 \frac{\frac{y}{np \vdash \boxed{np}} \quad \frac{\alpha}{\boxed{s} \vdash s}}{\boxed{(np \setminus s)} \vdash (np \cdot \setminus \cdot s)} (\setminus L) \\
 \frac{\boxed{(np \setminus s)} \vdash (np \cdot \setminus \cdot s)}{(np \setminus s) \vdash (np \cdot \setminus \cdot s)} \swarrow \\
 \frac{(np \setminus s) \vdash (np \cdot \setminus \cdot s)}{(np \cdot \otimes \cdot (np \setminus s)) \vdash s} \\
 \frac{np \vdash (s \cdot / \cdot (np \setminus s))}{\boxed{np} \vdash (s \cdot / \cdot (np \setminus s))} \nwarrow \quad \frac{[\text{teacher}]}{n \vdash \boxed{n}} \\
 \frac{\boxed{np} \vdash (s \cdot / \cdot (np \setminus s)) \quad n \vdash \boxed{n}}{\boxed{(np/n)} \vdash ((s \cdot / \cdot (np \setminus s)) \cdot / \cdot n)} (/L) \\
 \frac{\boxed{(np/n)} \vdash ((s \cdot / \cdot (np \setminus s)) \cdot / \cdot n)}{(np/n) \vdash ((s \cdot / \cdot (np \setminus s)) \cdot / \cdot n)} \swarrow \\
 \frac{(np/n) \vdash ((s \cdot / \cdot (np \setminus s)) \cdot / \cdot n)}{((np/n) \cdot \otimes \cdot n) \vdash (s \cdot / \cdot (np \setminus s))} \\
 \frac{((np/n) \cdot \otimes \cdot n) \vdash (s \cdot / \cdot (np \setminus s))}{(((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np \setminus s)) \vdash s} \\
 \frac{(((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np \setminus s)) \vdash s}{(((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np \setminus s)) \vdash \boxed{s}} \multimap
 \end{array}$$

$$[\dagger] = \lambda\alpha.([\text{some}]^{\lceil np/n \rceil^\perp} \langle \lambda y.([\text{left}]^{\lceil np \setminus s \rceil^\perp} \langle y^{np}, \alpha^{s^\perp} \rangle), [\text{teacher}]^n \rangle) \quad :: \quad s^{\perp\perp}$$

Lexical semantics: from MILL to IL



For $[\cdot]$ the interpretation of lexical constants is a ‘black box’.

The $(\cdot)^\ell$ map unpacks the lexical constants.

- ▶ constants are assigned a λ term of type $([\cdot]^{(\perp)})^\ell$
- ▶ the $\cdot^\ell$ translation is an IL term: non-linear recipes are allowed

Types Target atoms e (entities), t (booleans). Response type interpreted as t

$$np^\ell = e \quad ; \quad n^\ell = e \rightarrow t \quad ; \quad s^\ell = \perp^\ell = t$$

Complex types: replace linear by corresponding IL operations:

$$(A \otimes B)^\ell = A^\ell \times B^\ell \quad ; \quad (A \multimap B)^\ell = A^\ell \rightarrow B^\ell$$

Lexical sample Some lexical entries given bias $s : -, np, n : +$.

You can convince yourself that the given terms are welltyped as $\{\cdot\}^\ell$.

SOURCE TYPE	CONSTANT	$\{\cdot\}^\ell$
$np \backslash s$	left	$\lambda \langle x, c \rangle. (c \text{ (LEFT}^{et} x))$
$(np \backslash s) / np$	likes	$\lambda \langle \langle x, c \rangle, y \rangle. (c \text{ (LIKE}^{et} y x))$
$(np / n) \otimes n$	everyone	$\langle \lambda \langle x, y \rangle. (\forall \lambda z. (\Rightarrow (y z) (x z))) \rangle, \text{PERSON}^{et} \rangle$
np / n	some	$\lambda \langle x, y \rangle. (\exists \lambda z. (\wedge (y z) (x z)))$
n	girl	GIRL^{et}
$(np \backslash s) / s$	thinks	$\lambda \langle \langle x, c \rangle, q \rangle. (c \text{ (THINK}^{((tt)t)et} q x))$
$s / (np \backslash s)$	everybody	$\lambda \langle c, v \rangle. (c \text{ (}\forall \lambda x. (v \langle x, id \rangle)))$

Notation: $\{\cdot\}$ is $\lceil \cdot \rceil$ for positive types, $\lceil \cdot \rceil^\perp$ for negatives.

$\lambda \langle x, y \rangle. M$ is $\lambda z. M[x/\pi_0 z, y/\pi_1 z]$ (π_0, π_1 first/second projection of pair term);

id : identity function $\lambda p^t. p$.

Scope construal

- ▶ local construal: 2 interpretations for “every student likes some teacher”

$\lambda\alpha.([\text{every}] \langle \lambda y.([\text{some}] \langle \lambda z.([\text{likes}] \langle \langle y, \alpha \rangle, z \rangle), [\text{teacher}] \rangle), [\text{student}] \rangle)$

$\lambda\alpha.(\forall \lambda z.((\Rightarrow (\text{STUDENT } z)) (\exists \lambda y.((\wedge (\text{TEACHER } y)) (\alpha ((\text{LIKES } y) z))))))$

$\lambda\alpha.([\text{some}] \langle \lambda y.([\text{every}] \langle \lambda z.([\text{likes}] \langle \langle z, \alpha \rangle, y \rangle), [\text{student}] \rangle), [\text{teacher}] \rangle)$

$\lambda\alpha.(\exists \lambda z.((\wedge (\text{TEACHER } z)) (\forall \lambda y.((\Rightarrow (\text{STUDENT } y)) (\alpha ((\text{LIKES } z) y))))))$

- ▶ non-local construal: 2 interpretations for “Alice thinks some student left”

$\lambda\alpha.([\text{thinks}] \langle \langle [\text{alice}], \alpha \rangle, \lambda\beta.([\text{some}] \langle \lambda z.([\text{left}] \langle z, \beta \rangle), [\text{student}] \rangle) \rangle)$

$\lambda\alpha.(\alpha ((\text{THINKS } \lambda\beta.(\exists \lambda x.((\wedge (\text{STUDENT } x)) (\beta (\text{LEFT } x)))) \text{ ALICE})))$

$\lambda\alpha.([\text{some}] \langle \lambda y.([\text{thinks}] \langle \langle [\text{alice}], \alpha \rangle, \lambda\gamma.([\text{left}] \langle y, \gamma \rangle) \rangle), [\text{student}] \rangle)$

$\lambda\alpha.(\exists \lambda z.((\wedge (\text{STUDENT } z)) (\alpha ((\text{THINKS } \lambda\gamma.(\gamma (\text{LEFT } z))) \text{ ALICE}))))$

polarity bias $s : -, np, n : +$

Summarizing

Focusing regime + attendant continuation semantics optimizes syn/sem interface:

- ▶ ☺ determiners are uniformly typed as np/n ; desired sem type $(et)(et)t$ results from CPS translation
- ▶ ☹ compare: syntactic complications $(s/(np \setminus s))/n$ etc motivated by purely semantic considerations; ad hoc extra typings or structural rules

To do How to delimit non-local construal?

- ▶ “Alice thinks *some* student left” has a $\exists > \text{THINKS}$ interpretation; “Alice thinks *every* student left” only $\text{THINKS} > \forall$
- ▶ imposing locality by means of $\diamond, \square$ (cf Part 1) won't do: the unary modalities are transparent for residuation inferences

More To Do ...

The scope construal case study is couched entirely within the **NL** fragment of **LG**.

Are there convincing uses of the Grishin connectives/linear distributivities in linguistics?

- ▶ Moot (2008) uses **LG** to simulate the adjunction operation of TAG, a key example of a mildly context-sensitive formalism; but
- ▶ Melissen (2009): **LG** has recognizing capacity beyond MCS
- ▶ apart from these formal grammar studies, are there actual NLP uses?