Exponentiation and Iteration in the Lambek Calculus and Its Variants

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2. **L** is one of the **substructural logics** (see [Došen 1992; Restall 2000]): in the sequent form, it has the same **logical** rules as **Int**, but lacks **structural** rules: weakening, contraction, permutation.

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2. **L** is one of the **substructural logics** (see [Došen 1992; Restall 2000]): in the sequent form, it has the same **logical** rules as **Int**, but lacks **structural** rules: weakening, contraction, permutation.
3. As **Int** is the logic of Heyting lattices, the Lambek calculus (extended with lattice-theoretic, so-called **additive** operations) is the logic of a broader class—**residuated lattices** (see [Galatos, Jipsen, Kowalski, Ono 2007]).

The Lambek Calculus

4. Another natural class of residuated lattices (besides algebras of formal languages) are formed by algebras on **binary relations;** see completeness results [Andréka, Mikulás 1994]. This corresponds to interpreting Lambek formulae as **actions** (e.g., in a computational system). Extending the language with **Kleene star** [Kleene 1956] yields **action logic** [Pratt 1991; Kozen 1994] and its infinitary version [Buszkowski, Palka 2007].

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5. Finally, the Lambek calculus can be viewed as a version of non-commutative intuitionistic **linear logic** [Girard 1987; Abrusci 1990], with its informal resource interpretation. This enables adding **exponential** and **subexponential** modalities to the Lambek calculus.

The Lambek Calculus

$$\frac{}{A \to A}\ \mathsf{Id}$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}\ \cdot\mathsf{L} \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}\ \cdot\mathsf{R}$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, A \setminus B, \Delta \to C}\ \setminus\mathsf{L} \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ \setminus\mathsf{R}$$

$$\frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, B \,/\, A, \Pi, \Delta \to C}\ /\mathsf{L} \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A}\ /\mathsf{R}$$

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- The Cut rule is eliminable.
- We allow empty left-hand sides of sequents.

Lambek Grammars

N,	$(N \setminus S) \,/\, N$,	N	$\rightarrow S$
John	likes	Mary	

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$$N, \qquad (N \setminus S) / N, \qquad N \qquad \rightarrow S$$

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$$N / CN, \quad CN, \quad (CN \setminus CN) / (S / N), \quad N, \quad (N \setminus S) / N, \quad (N \setminus S) / N, \quad N \quad \rightarrow S$$

The girl whom John likes hates John

Lambek Grammars

$$\begin{array}{llll} N, & (N \setminus S) \,/\, N, & N & \to S \\ \text{John} & \text{likes} & \text{Mary} & \end{array}$$

- Here only \L and /L are used.

$$\begin{array}{llllllll} N \,/\, CN, & CN, & (CN \setminus CN) \,/\, (S \,/\, N), & N, & (N \setminus S) \,/\, N, & (N \setminus S) \,/\, N, & N & \to S \\ \text{The} & \text{girl} & \text{whom} & \text{John} & \text{likes} & \text{hates} & \text{John} & \end{array}$$

- Here we also use /R, to show that "John likes" is of type $(S \,/\, N)$.

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the girl whom John met yesterday $\quad \ldots \to N$

- This is a more complicated phrase, for which an extension of the Lambek calculus is used.

Models for the Lambek Calculus

- The Lambek calculus is the algebraic logic of **residuated monoids**, which a partially ordered monoids with residuals:

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- Lingustic applications motivate a concrete family of residuated monoids, whose elements are formal languages over an alphabet Σ, where

$$A \cdot B = \{uv \mid u \in A, v \in B\}$$
$$A \setminus B = \{u \in \Sigma^* \mid A \cdot \{u\} \subseteq B\}$$
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- Completeness was proved by Pentus [1998].

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- These are viewed as non-deterministic actions in a computing system.
- Product is relational composition, and divisions are "conditional actions":

$$R \cdot S = R \circ S = \{(x, z) \in W \times W \mid \exists y \in W\,((x, y) \in R \text{ and } (y, z) \in S)\}$$
$$R \setminus S = \{(y, z) \in W \times W \mid R \circ \{(y, z)\} \subseteq S\}$$
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- Completeness proved by Andréka and Mikulás [1994].
- Notice that completeness (for both classes of models) quickly ruins when one extends the set of operations!

Additive Operations and Constants

MALC, the **multiplicative-additive Lambek calculus**, is obtained from **L** by means of additive conjunction $\wedge$, additive disjunction $\vee$, and constants $\mathbf{0}$ and $\mathbf{1}$.

$$\frac{\Gamma, A, \Delta \to C \quad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C}\ \vee\mathsf{L} \qquad \frac{\Pi \to A}{\Pi \to A \vee B}\ \frac{\Pi \to B}{\Pi \to A \vee B}\ \vee\mathsf{R}$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C}\ \frac{\Gamma, B, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C}\ \wedge\mathsf{L} \qquad \frac{\Pi \to A \quad \Pi \to B}{\Pi \to A \wedge B}\ \wedge\mathsf{R}$$

$$\frac{}{\Gamma, \mathbf{0}, \Delta \to C}\ \mathbf{0}\mathsf{L} \qquad \frac{\Gamma, \Delta \to C}{\Gamma, \mathbf{1}, \Delta \to C}\ \mathbf{1}\mathsf{L} \qquad \frac{}{\to \mathbf{1}}\ \mathbf{1}\mathsf{R}$$

Exponential and Subexponentials

- We further extend **MALC** by means of linear logic **exponential** modality, yielding **!MALC**:

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !A, \Delta \to C}\ !\mathsf{L} \qquad \frac{!A_1, \ldots, !A_n \to B}{!A_1, \ldots, !A_n \to !B}\ !\mathsf{R} \qquad \frac{\Gamma, \Delta \to C}{\Gamma, !A, \Delta \to C}\ !\mathsf{W}$$

$$\frac{\Gamma, B, !A, \Delta \to C}{\Gamma, !A, B, \Delta \to C} \quad \frac{\Gamma, !A, B, \Delta \to C}{\Gamma, B, !A, \Delta \to C}\ !\mathsf{P} \qquad \frac{\Gamma, !A, !A, \Delta \to C}{\Gamma, !A, \Delta \to C}\ !\mathsf{C}$$

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- A more fine-grained structural control is given by **subexponentials**, in a polymodal system of $!^s$ indexed by $s \in \mathscr{I}$, where $\mathscr{I}$ bears a partial order $\preceq$.

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- A more fine-grained structural control is given by **subexponentials**, in a polymodal system of $!^s$ indexed by $s \in \mathcal{I}$, where $\mathcal{I}$ bears a partial order $\preceq$.
- Introduction rules:

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !^s A, \Delta \to C}\ !\mathsf{L} \qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \to B}{!^{s_1} A_1, \ldots, !^{s_n} A_n \to !^s B}\ !\mathsf{R},\ s_i \succeq s$$

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- These sets control which structural rules are allowed:

$$\frac{\Gamma, \Delta \to C}{\Gamma, !^{w}A, \Delta \to C}\ !\mathsf{W},\ w \in \mathscr{W} \qquad \frac{\Gamma, B, !^{e}A, \Delta \to C}{\Gamma, !^{e}A, B, \Delta \to C} \quad \frac{\Gamma, !^{e}A, B, \Delta \to C}{\Gamma, B, !^{e}A, \Delta \to C}\ !\mathsf{P},\ e \in \mathscr{E}$$

$$\frac{\Gamma, !^{c}A, \Phi, !^{c}A, \Delta \to C}{\Gamma, !^{c}A, \Phi, \Delta \to C} \quad \frac{\Gamma, !^{c}A, \Phi, !^{c}A, \Delta \to C}{\Gamma, \Phi, !^{c}A, \Delta \to C}\ !\mathsf{NC},\ c \in \mathscr{C}$$

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- Notice that contaction here appears in its **non-local form**, otherwise cut-elimination fails for $c \in \mathscr{C} - \mathscr{E}$
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  [Bayu Surarso, Ono 1996; Kanovich, K., Nigam, Scedrov 2018].
- Since !W and !NC simulate !P, we postulate $\mathscr{W} \cap \mathscr{C} \subseteq \mathscr{E}$.

Subexponentials in Lambek Grammars

- The permutation modality ($e \in \mathscr{E}$, $e \notin \mathscr{W} \cap \mathscr{C}$) is used for modelling **medial extraction**:

$N \,/\, CN,$	$CN,$	$(CN \setminus CN) \,/\, (S \,/\, !^{e}N),$	$N,$	$(N \setminus S) \,/\, N,$	$(N \setminus S) \setminus (N \setminus S)$	$\to N$
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- Notice that weakening is linguistically inacceptable, so we have the so-called **relevant modality**: $r \in \mathscr{C} \cap \mathscr{E}$, $r \notin \mathscr{W}$.
- In real linguistic applications, even more sophisticated, semi-non-associative, systems of structural control are used [Morrill 1992; Moortgat 1996].

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- Derivability in **MALC** is also decidable, being PSPACE-complete [Kanovich 1994; ...]
- In contrast, **!MALC** is undecidable (Σ^0_1-complete).
- The reason is undecidability of **entailment** from finite sets of hypotheses. The latter is encoded via **modalised deduction theorem**:

$$\to A_1, \dots, \to A_n \vdash_{\mathbf{MALC}} \Pi \to C \iff \vdash_{\mathbf{!MALC}} !A_1, \dots, !A_n, \Pi \to C.$$

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- Therefore, the fragment of **!MALC** with only $\backslash$ and ! is already undecidable.

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- With **local contraction**, cut-elimination can be restored in two ways. One either restricts the right rule

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !A, \Delta \to C}\ !\mathsf{L} \qquad \frac{!A \to B}{!A \to !B}\ !_1\mathsf{R} \qquad \frac{\Gamma, !A, !A, \Delta \to C}{\Gamma, !A, \Delta \to C}\ !\mathsf{C}$$

or allows contraction of sequences of !-formulae:

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !A, \Delta \to C}\ !\mathsf{L} \qquad \frac{!\Pi \to B}{!\Pi \to !B}\ !\mathsf{R} \qquad \frac{\Gamma, !\Pi, !\Pi, \Delta \to C}{\Gamma, !\Pi, \Delta \to C}\ !\mathsf{MC}$$

(Here if $\Pi = A_1, \ldots, A_n$, then $!\Pi = !A_1, \ldots, !A_n$.)

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- Namely, ! is interpreted as a monotone idempotent operation on binary relations which maps any relation to its **dense** subrelation.
- For the second system, ! should also commute with product: $!R \circ !S \subseteq !(R \circ S)$.

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- **Positive iteration** ("Kleene plus") is defined: $A^+ = A \cdot A^*$.

(Infinitary) Action Logic

- The algebraic logic of $*$-continuous action lattices is **infinitary action logic** $\mathbf{ACT}_\omega$ [Palka 2007], obtained from **MALC** by the following rules:

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  - Notice that Cut is eliminable in $\mathbf{ACT}_\omega$, but not in **ACT**.

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- Now the grammar is total iff the following sequent is derivable in $\mathbf{ACT}_\omega$:

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- For proving undecidability of the weaker system **ACT**, we introduce the notion of **regular totality** for context-free grammars, which captures **circular runs** of Turing machines.
- A grammar G is **regularly total**, if it includes the following rules:

$$S \Rightarrow aYT \qquad T \Rightarrow a \qquad T \Rightarrow aT,$$

and the following holds: (1) $Y \Rightarrow^* w$ for each w with $|w| = n$; (2) $S \Rightarrow^* w$ for each w with $|w| \leqslant n + 1$.

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- For a regularly total $G_{\mathtt{M}}$, the sequent $E^+ \to S$ is provable already in **ACT**.
- Unfortunately, the converse does not hold: there could be cases where $E^+ \to S$ is derivable in **ACT**, but $\mathtt{M}$ does not loop.

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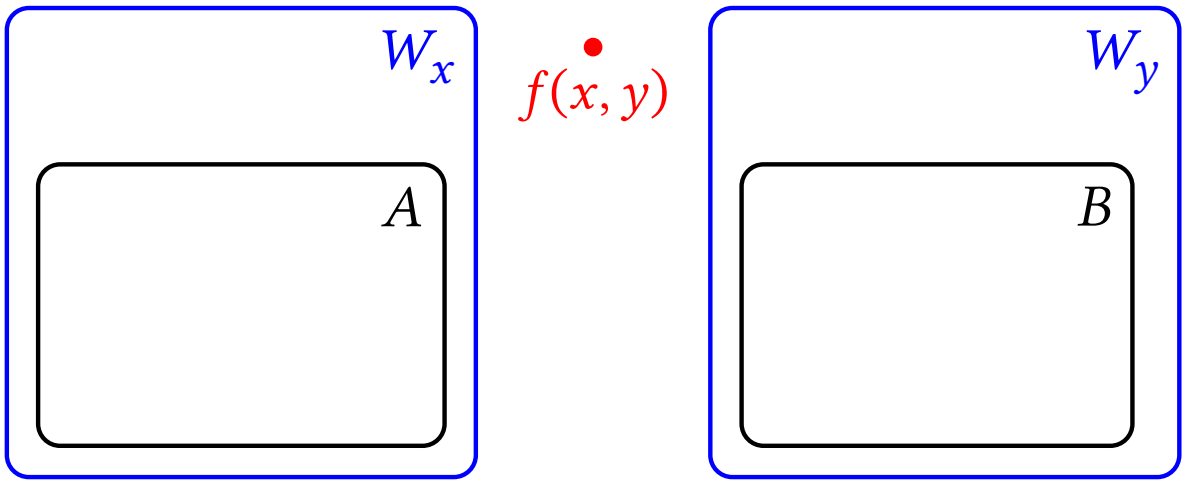

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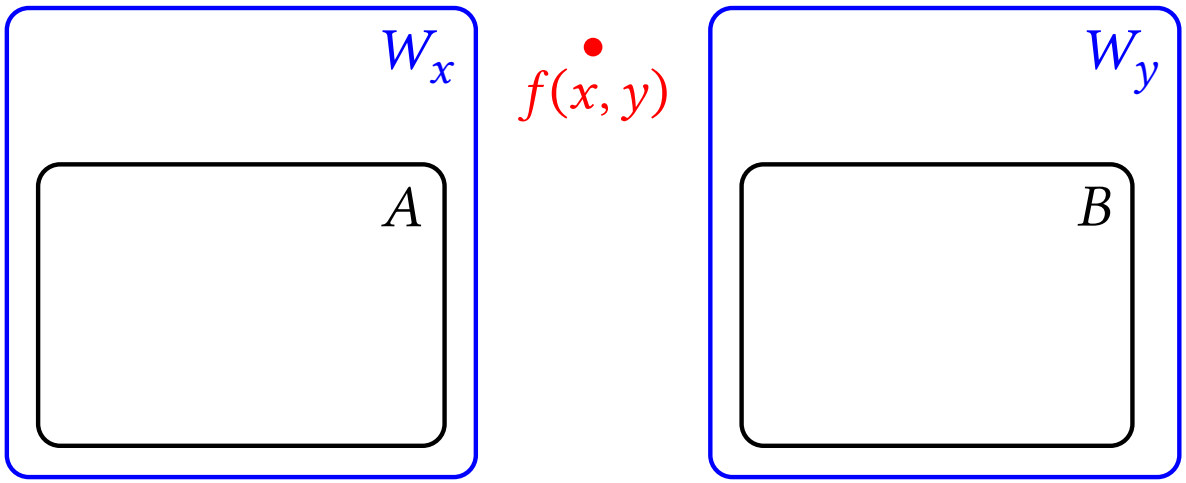

- f effectively prevents A and B from being separated by a decidable set.

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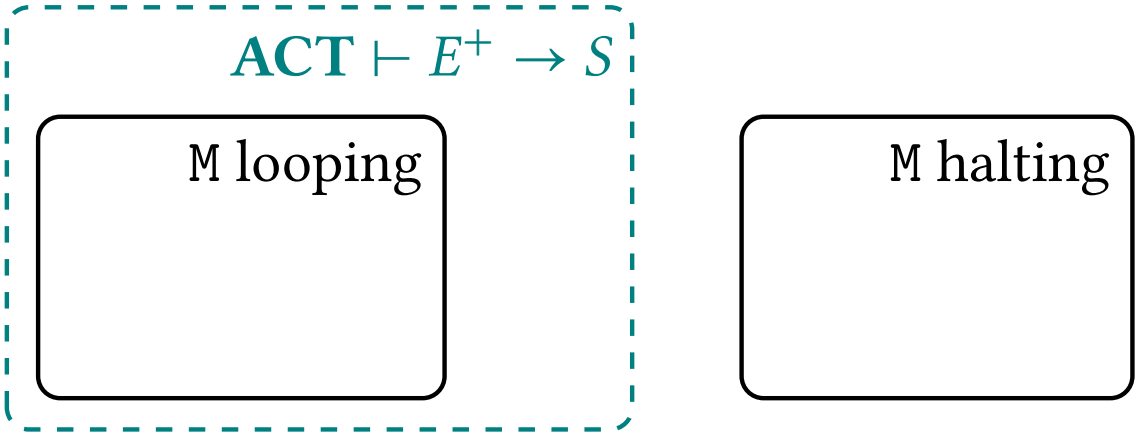

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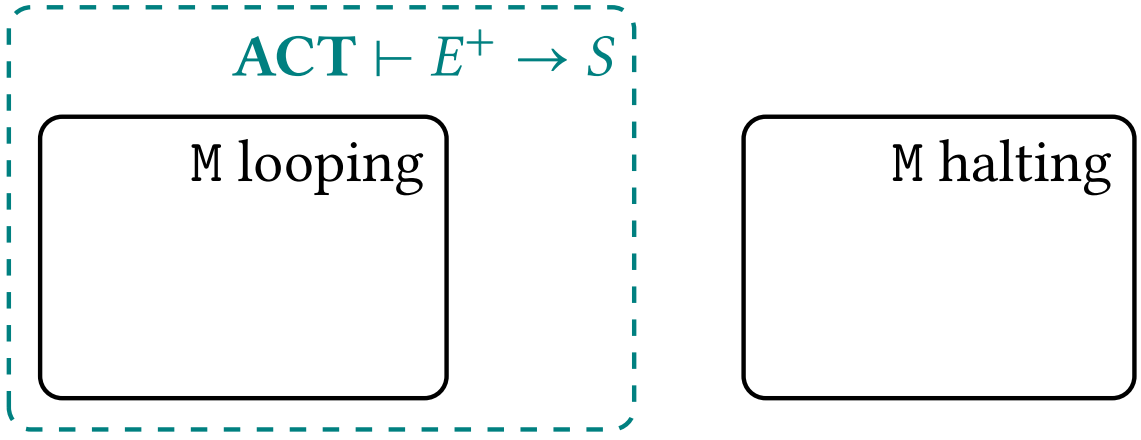

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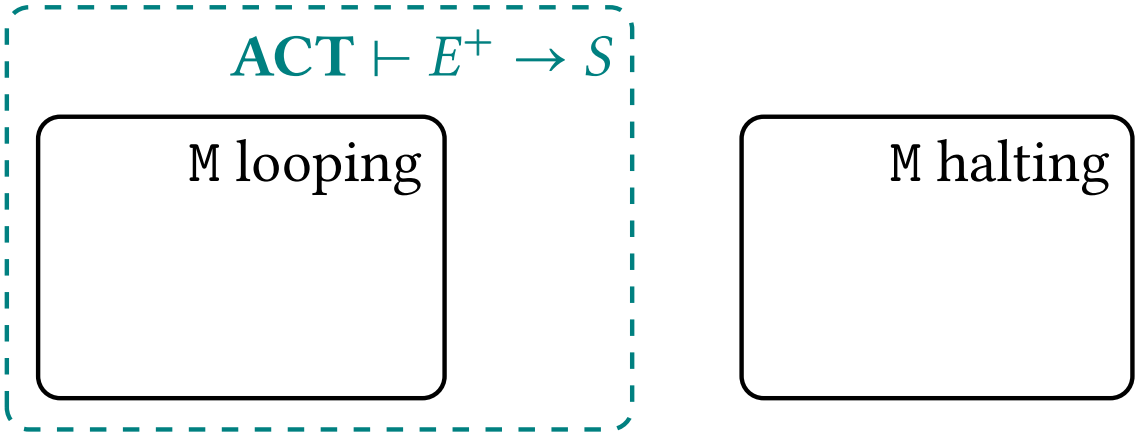

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- This proves Σ^0_1-completeness of **ACT** [K. 2021].

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- Next we, again, internalise reasoning from hypotheses into $\mathbf{!ACT}_\omega$ using modalised deduction theorem.

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- Well-foundedness of R is equivalent to well-foundedness (e.g., correctness) of the proof of $t, b^* \to r$ from the given set of hypotheses in $\mathbf{ACT}_\omega$.

Iterative Divisions

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- We shall need this result later on.

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- The lower bound is proved via encoding truth of computable infinitary propositional formulae of rank $< \omega^\omega$ [Ash, Knight 2000; Montalbán 2022].
- As a matter of fact, a similar technique applied to $\mathbf{ACT}_\omega$ computes Π^0_1-formulae defining derivability for sequents of given ordinal measure. This gives an alternative Π^0_1 upper bound proof.

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- From this, using strong completeness and complexity results, one derives Π^1_1-completeness for the equational theory of formal language algebras, in the language of $\backslash$, $\backslash\backslash$, $\wedge$, $\mathbf{1}$.

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- Thus, we get Π^1_1-completeness for the equational theory of algebras of binary relations, in the language of $\backslash$, $\backslash\backslash$, $\wedge$, $\mathbf{0}$, $\mathbf{1}$.
  - This solves an open problem by Buszkowski [1982], who conjectures Π^0_1-completeness.

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- In **NL** without additives and in the **distributive** version of **NL** with additives, entailment from finite sets of hypotheses is **decidable** [Buszkowski, Farulewski 2009].

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- In the associative setting, in contrast, distributivity is usually an obstacle.
- In particular, we do not know exact complexity of distributive versions of **MALC** and $\mathbf{ACT}_\omega$.
- Decidability of distributive **MALC** shown by Kozak [2009].

Thanks

Thanks!*

- This talk is partially based on joint work with Eben Blaisdell, Max Kanovich, Glyn Morrill, Vivek Nigam, Elaine Pimentel, Tikhon Pshenitsyn, Andre Scedrov, and Stanislav Speranski, all of whom the author is indebted to.
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