Morita Equivalence for Kleene Algebras

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Kleene Algebras and Modules

Kleene Algebras

Definition (Kozen)

An algebra $\mathfrak{K} = \langle K, +, \cdot, 0, 1, * \rangle$ of signature $\langle 2, 2, 0, 0, 1 \rangle$ is a *Kleene algebra* when

1. The reduct $\langle K, +, \cdot, 0, 1 \rangle$ is an additively-idempotent semiring (so $x + x = k$ for every $x \in K$);
2. for every $x \in K$, $x^* = 1 + xx^* = x^*x + 1$,
3. for every $a, b, x \in K$, if $b + ax \leq x$ then $a^*b \leq x$,
4. for every $a, b, x \in K$, if $b + xa \leq x$ then $ba^* \leq x$.

For the last two conditions, $\leq$ is the lattice order for $\langle K, + \rangle$, defined by $x \leq y$ precisely when $x + y = y$. The unary operation $*$ is called the *asterate*.

Examples of Kleene Algebras

1. The algebra of regular languages in a fixed alphabet,
2. The algebra of relations on a fixed set, where R^* is interpreted as the reflexive transitive closure of the relation R.
3. The algebra of sequential processes over a fixed collection of primitive operations.
4. The tropical algebra $\langle \mathbf{R}^{\geq 0} \cup \{\infty\}, \min, +, \infty, 0, *\rangle$, where $r^* = 0$ for $r \in \mathbf{R}^{\geq 0}$.

Kleene Modules

- Modules are central to the study of rings.
- The analogous statement is not true for Kleene algebras, but perhaps the subject is moving in that direction.

Kleene Modules

Definition

For K a Kleene algebra, an additively-idempotent (left) semiring module $\langle M, +, 0 \cdot : K \times M \to M \rangle$ is a *Kleene module* when it satisfies the following for $a \in K$ and $m \in M$: if $am \leq m$ then $a^* m \leq m$.
Right Kleene modules and (K, S)-bimodules for K and S Kleene algebras are defined similarly.

Examples of Kleene Modules

1. The endomorphisms $\mathrm{End}(K)$ of a Kleene algebra K form a left K-module.
2. The Lindenbaum-Tarski algebra of a propositional dynamic logic, with $p[\phi] = [\langle p \rangle \phi]$ for ϕ a sentence in the propositional dynamic logic, $[\cdot]$ denoting the Lindenbaum-Tarski class of a sentence, and p an element of the process Kleene algebra.
3. For any Kleene K-module M, $\mathrm{End}(M)$ is a K-module.

In particular the matrix modules $\mathrm{End}(K^n)$ for $n \in \mathbf{N}$ are K-modules (in fact, (K, K)-bimodules).

Representations and the Morita category

Kleene Algebra Representations

In ring theory representations are another view of modules and provide another fundamental perspective on rings. This carries over to Kleene algebras:

Definition

Let K be a Kleene algebra and $\langle M, +, 0\rangle$ be a commutative idempotent monoid. A (left) *representation* of K over M is a semiring homomorphism $\phi : K \to \mathrm{End}(M)$ with the property that for any $m \in M$ and $a \in A$, if $\phi(a)(m) \leq m$ then $\phi(a^*)(m) \leq m$.

It is easy to see that this is just another formulation of Kleene modules. In particular, if two Kleene algebras have equivalent categories of modules, then they have the same representation theory.

The Morita Category

The Morita category of rings has rings as objects, (R, S)-bimodules as morphisms from R to S, bimodules ${}_R R_R$ as identities, and tensor product as composition.

Definition

The *Morita category of Kleene algebras* has

- Kleene algebras as objects,
- (K, S)-bimodules as morphisms from K to S,
- bimodules ${}_K K_K$ as identities,
- tensor product of Kleene modules as composition.

In order for this definition to make sense, one must verify that a tensor product exists for Kleene modules and has the expected properties. The usual arguments for rings carry over without special difficulties.

Morita Equivalence

Definition

Kleene algebras K and S are *Morita-equivalent* if and only if they are isomorphic in the Morita category.

This means that there are Kleene modules ${}_K M_S$ and ${}_S N_K$ such that

- ${}_K M_S \otimes {}_S N_K \cong {}_K K_K$, and
- ${}_S N_K \otimes {}_K M_S \cong {}_S S_S$.

Representations and Morita Equivalence

Morita-equivalent Kleene algebras K and S have the same representation theory, as we can use the components of an equivalence to change scalars:

- ${}_K P \mapsto {}_S N_K \otimes {}_K P$, and
- ${}_S P \mapsto {}_K M_S \otimes {}_S P$.

These maps are easily seen to be inverses.
This also enables us to see that Kleene algebras are Morita-equivalent if and only if they have equivalent categories of (left) modules.

Matrices and Morita Equivalence

If Kleene algebras K and S are Morita-equivalent as semirings then this is witnessed by a pair of inverse Kleene-bimodules and so they are in fact Morita equivalent as Kleene algebras. Hence the proof of a matrix characterization of Morita equivalence carries over from rings as well:

Theorem

Kleene algebras K and S are Morita-equivalent if and only if for some dimension n and full idempotent e in the semiring M of $n \times n$ matrices over (without loss of generality) K, $S \cong eMe$ (with the matrix asterate due to Kozen).

Here an idempotent $e \in M$ is *full* precisely when $MeM = M$.
In particular, when two Kleene algebras are Morita-equivalent, one may be viewed as a matrix algebra over the other.

Automata and Morita Equivalence

- A (finite-state) automaton over a Kleene algebra is simply an $n \times n$ matrix with entries in K, where n is the number of states and matrix elements represent transitions.
- Morphisms between automata are bisimulations, which are relations between states that preserve transitions.
- Because in a pair of Morita-equivalent Kleene algebras, one algebra is a matrix algebra over the other, Morita-equivalent Kleene algebras have equivalent categories of automata.
- Unfortunately, the reverse is not true.

Applications of Morita Equivalence?

- We have already seen that Morita equivalence implies equivalence of representation theory and of categories of automata.
- Given the Lindenbaum-Tarski algebra of a propositional dynamic logic (not necessarily classical!) with modalities from a Kleene algebra K, the algebra K may be equivalently replaced by any Morita-equivalent algebra S. In particular, using matrices in propositional dynamic logic adds no expressive power.
- (Co)homology of Kleene algebrae? Cohomology of semirings has been studied to a degree, and may yield insights about Kleene algebrae.

Tensor Algebras

The Tensor Algebra of a Semiring-module

Definition

Let M be a semiring-bimodule over a semiring S. The *tensor algebra* of M is defined to be

$$T = \bigoplus_{k=0}^{\infty} M^{\otimes k},$$

where $M^{\otimes k} = M \otimes M \cdots \otimes M$ is the k-fold tensor product of M. T naturally has the structure of a semiring.

The Tensor Algebra of a Kleene-module

Let M be a (K, K)-bimodule where K is a Kleene algebra.

- The tensor algebra T of M is an additively idempotent semiring, but we would like a Kleene algebra.
- If K is $*$-continuous, we can fix this by taking the $*$-completion of T, which is adjoint to the forgetful functor from Kleene algebras to additively-idempotent semirings.
- Is there an extension to a Kleene algebra in general?

Thus we have a means of canonically extending an arbitrary Kleene module over a $*$-continuous Kleene algebra to a Kleene algebra, via the tensor product.

Applications of Tensor Kleene Algebras?

- If Lindenbaum-Tarski modules for logics of processes from an algebra K and tests from a different kind of algebra B (Boolean, Heyting, MV, etc.) were naturally (K, K)-bimodules, then the tensor Kleene algebra would provide a way of canonically extending such a module to a Kleene algebra with generalized tests. Unfortunately, Lindenbaum-Tarski modules are only obviously (K, K)-bimodules when K is commutative, so more work is needed to make this perspective useful.
- Automata over a Kleene algebra K may be extended to tensor algebras, and the result includes intriguing examples of multidimensional "automata" which may warrant further exploration.

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