How to extend the set-theoretic vocabulary in a type-safe way?

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Basic language of NF(U)

Quine's NF: a very elegant theory of classes,
intended as a better foundation of mathematics.
Axioms: only extensionality and stratified comprehension.
Problems: (Specker) NF $\models \neg$AC; no consistency proof

Jensen's **NFU**: a product of observation that a
consistency proof can be obtained if atoms are allowed.
Basic vocabulary σ_0: three relation symbols
(binary $=$, binary $\in$, unary set), no function and
no constant symbols (therefore terms are just variables)

That gives us atomic formulas (over σ_0): we get general formulas by using logical constants (basic $\bot$; derived $\top$), connectives (basic $\rightarrow$; derived $\neg$, $\vee$, $\wedge$, $\leftrightarrow$) and quantifiers (basic $\forall$; derived $\exists$, $\nexists$, $\exists!$, restricted quantification; later, quantification over terms)

Connectedness

Two nonconstant terms in a formula φ are 1-connected if

1. both of them are arguments of some atomic formula $R(t_1, \dots, t_k)$ [they are t_i and t_j for some $i < j$], or
2. one of them is an argument of another [they are t_i and $f(t_1, \dots, t_k)$ for some i and function symbol f].

[The rule (2) is currently not applicable (no function symbols), but it will apply later. Currently, (variables) x and y are 1-connected if and only if $x \in y$ or $x = y$ is a subformula of φ.]

Connectedness is the equivalence closure of 1-connectedness.
A formula is connected if all its nonconstant terms are connected.

We will only consider connected formulas.

Thesis: All useful formulas, and all formulas actually appearing in developing some theory, are connected.

Stratification

Each k-ary function symbol has a *signature*: a k-tuple of integers. Each relation symbol R also has a signature $[R]$, starting with 0. Currently, $[\mathsf{set}] := (0)$, $[=] := (0, 0)$ and $[\in] := (0, 1)$.

Stratification of a formula φ is a mapping tp from all nonconstant terms of φ to integers, satisfying the following conditions:

1. if $R(t_1, \dots, t_k)$ is a subformula of φ, and t_i and t_j are nonconstant terms, then $tp(t_i) - \delta_i = tp(t_j) - \delta_j$, where $[R] = (\delta_1, \dots, \delta_k)$ (and $\delta_1 = 0$).
2. if $t := f(t_1, \dots, t_k)$ is a subterm of φ and t_i is a nonconstant term (implying t is nonconstant too), then $tp(t_i) - \delta_i = tp(t)$.

[As on the previous slide, condition (2) applies only later.]

A formula is stratified if it has a stratification. We will only consider stratified (sentences, or) formulas with one distinguished free variable, which we call z: such a formula has a unique grounded stratification satisfying $tp(z) = 0$ (or any given number).

Introducing relation symbols

Let $\psi(z, \vec{x}^k)$ be a stratified formula (with exactly $k+1$ variables z, $\vec{x}$ free), and let tp be its grounded stratification. We introduce a new $(k+1)$-ary relation symbol R_ψ with signature $\big(0, tp(x_i)_{i=1}^k\big)$ and an axiom

$$\forall z \forall \vec{x} \big(R_\psi(z, \vec{x}) \leftrightarrow \psi\big).$$

This is a conservative extension.

Example

$\neq$ is $R_{\neg(z=x_1)}$. $\subseteq$ is $R_{(\forall u \in z)(u \in x_1)}$. $\subset$ is $R_{z \neq x_1 \wedge z \subseteq x_1}$.
$\ni$ is $R_{x_1 \in z}$. "being disjoint" is $R_{\neg(\exists u \in x_1)(u \in z)}$.
$[\neq] = [\subseteq] = [\subset] = [\text{disjointness}] = (0, 0)$, while $[\ni] = (0, -1)$.

Let $\text{func}(f, X, Y)$ mean $f : X \to Y$. Then $\text{func} = R_\psi$ for

$$\psi := (\forall u^{-3} \in x_1{}^{-2})(\exists! v^{-3} \in x_2{}^{-2})\big((u, v)^{-1} \in z^0\big) \wedge \\ \wedge (\forall p^{-1} \in z)(\exists u \in x_1)(\exists v \in x_2)\big(p = (u, v)\big),$$

with $[\text{func}] = (0, -2, -2)$. [Grounded stratification written above.]

Introducing function symbols

Let $\psi(z, \vec{x}^k)$ be a stratified formula (with exactly $k + 1 \geq 2$ variables z, $\vec{x}$ free), and let tp be its grounded stratification. We introduce a new k-ary function symbol f_ψ with signature $\big(tp(x_i) - 1\big)_{i=1}^{k}$ and an axiom (again, a conservative extension)

$$\forall \vec{x}\big(\mathrm{set}\big(f(\vec{x})\big) \wedge \forall z\,(z \in f(\vec{x}) \leftrightarrow \psi)\big).$$

Example

$\cup$ is $f_{z \in x_1 \vee z \in x_2}$, and $\setminus$ is $f_{z \in x_1 \wedge \neg(z \in x_2)}$, with $[\cup] = [\setminus] = (0, 0)$.
For $\iota(x) := \{x\}$, $\iota = f_{z = x_1}$ and $[\iota] = (-1)$.
Denote $up(x, y) := \iota(x) \cup \iota(y)$, with signature $(-1, -1)$.
For $\mathrm{pair}(x, y) := (x, y)$, $\mathrm{pair} = f_{z \in up(\iota(x), up(x, y))}$, $[\mathrm{pair}] = (-2, -2)$.
For $\mathrm{init.seg.}(X, \prec, x) := p_{(X, \prec)}(x)$, we have $[\mathrm{init.seg.}] = (0, 2, -1)$, since $\mathrm{init.seg.} = f_{z \in x_1 \wedge (z, x_3) \in x_2 \wedge \mathrm{poset}(x_1, x_2)}$.

[As usual, we prescribe that "undefined" terms denote $\emptyset$.]

Introducing constant symbols

Let $\psi(z)$ be a stratified formula with only z free. We introduce a new constant symbol c_ψ and an axiom (conservative extension)

$$\mathsf{set}(c_\psi) \wedge \forall z\,(z \in c_\psi \leftrightarrow \psi).$$

Constant symbols are not typed, and they don't have a signature.

Example

$\emptyset := c_{z \neq z}$, $V := c_{z=z}$, $SET := c_{\mathsf{set}(z)}$.
If we define $Sc := f_{(\exists u \in z)(z \setminus \iota(u) \in x_1)}$, $\bigcap := f_{(\exists v \in x_1)(z \in v)}$ and $IND(z) := \big(\iota(\emptyset) \in z \wedge (\forall u \in z)(Sc(u) \in z)\big)$, then $\mathbb{N} = \bigcap c_{IND}$.

And so on. We can define new symbols using stratified formulas over some layer σ_n, and put them in a new layer σ_{n+1}. In the limit (union), we get a **maximally extended language** $\sigma_\omega := \bigcup_{n \in \omega} \sigma_n$ and a theory $\mathcal{T}_\omega$ which is a *conservative* extension of the basic theory $\mathcal{T}_0$ over σ_0 (containing extensionality, sethood and stratified comprehension, but possibly also other axioms such as choice).

Reduction down to basic language

Moreover, using additional axioms, every formula φ over σ_ω can be converted, constructively, into an equivalent (in $\mathcal{T}_\omega$) formula φ_0 over σ_0, by systematically eliminating new symbols (from higher layers to lower ones). For start, any subformula $R_\psi(t, t_1, \dots, t_k)$ can be replaced by $\psi(t|z, t_1|x_1, \dots, t_k|x_k)$. Also, obviously, any formula of the form $t \in c_\psi$ can be replaced by $\psi(t|z)$, but also other formulas involving c_ψ, as shown:

$t = c_\psi$ rewrite as $\forall u\,(u \in t \leftrightarrow t \in c_\psi)$ by weak extensionality (new axiom ensures $\mathsf{set}(c_\psi)$), and then apply the above;

$c_\psi \in t$ rewrite as $(\exists u \in t)\,(u = c_\psi)$ by logical axioms of equality (Leibniz substitutability), and then apply the above.

The same process can be performed with function symbols.

φ is stratified if and only if φ_0 is stratified;
moreover, tp_φ and tp_{φ_0} agree on all free variables of both formulas.