On the Proof Complexity of Combinatorial Principles

(the role of kernelization)

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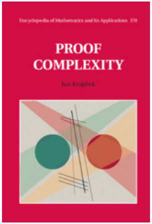
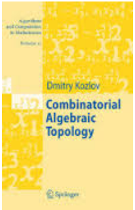
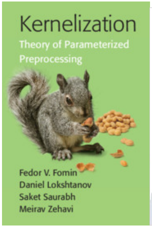
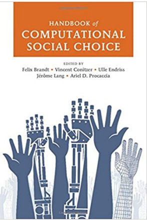
Take-home message





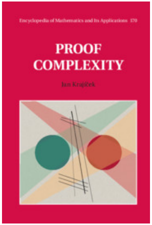




- Motivation: separating complexities of Frege and extended Frege proofs.
- Answer: notions from parameterized complexity can help us obtain efficient Frege proofs.
- Applications: various statements, including ones from combinatorial topology and computational social choice.

Caution: Emphasis on "the story", rather than technical details.

Reminder: Propositional proof complexity



- Proof systems for propositional unsatisfiability, e.g. resolution
- C or x, D or $\overline{x} \rightarrow (C$ or $D)$; $x, \overline{x} \rightarrow \square$.
- Complexity $=$ minimum length of a proof.

EXAMPLE: Pigeonhole principle tautologies PHP^n_{n-1} have $2^{\Omega(n)}$ resolution complexity.

$\boxed{n \text{ pigeons in } n-1 \text{ holes} \Rightarrow \text{at least two pigeons in same hole !}}$

Proof complexity of the pigeonhole principle

- Pigeonhole formula(s): PHP_n^{n-1}
- $X_{i,j} = 1$ "pigeon i goes to hole j".
- $X_{i,1}$ or $X_{i,2}$ or $\ldots$ or $X_{i,n-1}$, $1 \leq i \leq n$ (each pigeon goes to (at least) one hole)
- $\overline{X_{k,j}}$ or $\overline{X_{l,j}}$ (pigeons k and l do not go together to hole j).
- Resolution complexity: $2^{\Omega(n)}$! (Haken)

Buss (J. Symb. Logic): PHP_n^{n-1} has poly-size Frege proofs.

Frege proofs?

- @ boundaries of proof complexity: Frege proofs. For concreteness [Hilbert Ackermann]
  - propositional variables $p_1, p_2, \ldots.$, connectives $\neg$, or .
  - Axiom schematas:
    1. $\neg(A \text{ or } A) \text{ or } A$
    2. $\neg A \text{ or } (A \text{ or } B)$
    3. $\neg(A \text{ or } B) \text{ or } (B \text{ or } A)$
    4. $\neg(\neg A \text{ or } B) \text{ or } (\neg(C \text{ or } A) \text{ or } (C \text{ or } B))$
  - Rule: From A and $\neg A$ or B derive B.
- Other systems, sequent calculus (LK), etc.

All Frege proof systems equivalent (polynomially simulate eachother) S.A. Cook,R. Reckhow. "The relative efficiency of propositional proof systems." J. Symb. Logic 44.1(1979):36-50.

Frege versus extended Frege

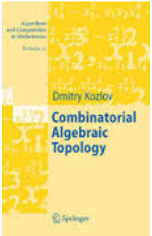
- extended Frege: Frege + variable substitutions $X \leftrightarrow \Phi(\overline{Y})$. Proves same formulas, perhaps more efficiently.

Open: Is extended Frege more powerful than Frege ?

Bonet, M.L., S. Buss, T. Pitassi. "Are there hard examples for Frege systems?." Feasible Mathematics II. Birkhauser, 1995. 30-56.

- Most natural formulas: (quasi)polynomial $(2^{log(n)^{O(1)}})$ Frege proofs.
- Some examples: "$(AB = I) \Rightarrow (BA = I)$" tautologies [Hrubeš, Tzameret CCC'2009], Paris-Harrington tautologies [Carlucci, Galesi, Lauria. CCC, 2011], Frankl Theorem [Buss et al. 2014].

Wishful thinking (around 2014)



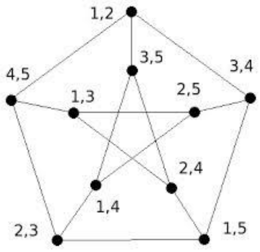
Perhaps translating into SAT a statement that is (mathematically) hard to prove yields a natural candidate for the separation.

- (Martin Kneser, Jaresbericht DMV 1955): Let $n \geq 2k-1 \geq 1$. Let $c : \binom{n}{k} \to [n-2k+1]$. Then there exist two disjoint sets A and B with $c(A) = c(B)$.
- $k = 1$: Pigeonhole principle !
- $k = 2, 3$: combinatorial proofs (Stahl, Garey & Johnson)
- $k \geq 4$: proved in 1977 (Lovász) using Algebraic Topology.

Combinatorial proofs (Matousek, Ziegler). "hide" Alg. Topology No "purely combinatorial" proof (was) known.

Kneser's Conjecture (II)

- the chromatic number of a certain graph $Kn_{n,k}$ (at least) $n - 2k + 2$. (exact value)
- Vertices: $\binom{n}{k}$. Edges: disjoint sets.
- E.g. $k = 2$, $n = 5$: Petersen's graph has chromatic number (at least) three.



First results (paper @ SAT conference)

- naïve encoding $X_{A,k} = TRUE$ iff A colored with color k. Extends encoding of PHP
- $X_{A,1}$ or $X_{A,2}$ or $\ldots$ or $X_{A,n-2k+1}$ "every set is colored with (at least) one color"
- $\overline{X_{A,j}}$ or $\overline{X_{B,j}}$ $(A \cap B = \emptyset)$ "no two disjoint sets are colored with the same color"
- $k = 1$: PHP (studied by Buss).

- $Kneser_n^k$ reduces to $Kneser_{n+2}^{k+1}$.
- $k = 2$: poly-size Frege proofs.
- $k = 3$: poly-size extended Frege proofs.

(paper @ ICALP ⇒ Information and Computation)

- For every $k \geq 3$ $Kneser_k$ poly-size extended Frege proofs, quasi-poly-size Frege proofs.
- For every $k \geq 1$ can reduce verification of $Kneser_k$ to that of a finite number of examples.

For every fixed k, $Kneser_k$ has combinatorial proofs.

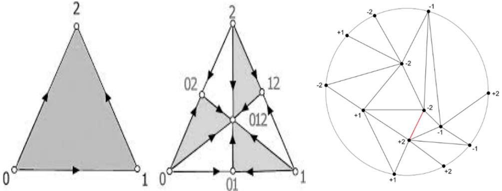
- Mathematically: Kneser follows from octahedral Tucker lemma (algebraic topology, exponential-size objects).
  - "Miniaturization" of this principle: truncated octahedral Tucker lemma.
  - class of propositional formulae, implies Kneser; candidates for separation.

Discrete version of Borsuk-Ulam: Octahedral Tucker's lemma

- Antipodally Symmetric Triangulation T of the n-ball. Barycentric subdivision, one vertex for each face

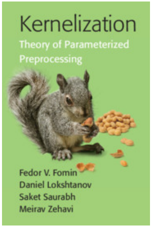
> - For any labeling of T with vertices from $\{\pm 1, \ldots, \pm(n-1)\}$ antipodal on the boundary there exist two adjacent vertices $v \sim w$ with $c(v) = -c(w)$.

- Intuition: no continuous (a.k.a simplicial) antipodal map from the n-ball to the n-sphere.



Second surprise: reverse-engineering proof of Kneser

(second paper @ICALP)



- For every $k \geq 1$ can "reduce" verifying an infinite number of examples to a finite number.
- Behind this type of reduction: kernelization.

Algorithmics, technique for preprocessing individual instances of a combinatorial problem.

Two-minute parameterized complexity

- Many problems in NP parameterized: instance size n, parameter k.
- Can get: complexity $O(n^k)$.
- Parameterized complexity: want complexity $O(f(k) \cdot poly(n))$.
- Kernelization: reduce instance (x, k) to "kernel instance" (x', k'), s.t. $(x, k) \in L$ iff $(x', k') \in L$ and

  $$\boxed{|x'|, k' \leq g(k) \text{ for some computable } g.}$$

- data reduction: algorithm A, maps (x, k) to (x', k') s.t. $(x, k) \in L$ iff $(x', k') \in L$ and $|x'| \leq |x|$, $k' \leq k$. Only for $|x| > g(k)$.
- algorithm: data reduction + bruteforce kernel instances.

- E.g. Vertex Cover: Given graph G and integer k, decide whether G has VC of size at most k. set of vertices that covers all edges.

Rule 1: v isolated vertex in G. G has VC of size k iff $G \setminus \{v\}$ has VC of size k.

Rule 2: v vertex in G, $deg(v) > k$. G has VC of size k iff $G \setminus (\{v\} \cup N(v))$ has VC of size $k - 1$.

THEOREM (parameterized complexity, informal): If G is a graph with more than k^2 vertices then one of Rules 1 and 2 can be applied.

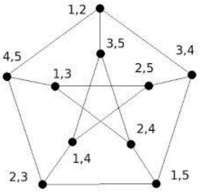
Main idea

- "Negative" instance (x, k) of parameterized problem in NP maps "canonically" to formula $\Phi(x, k) \in \overline{SAT}$.

- If Π_i proof for soundness of the i'th reduction step $(x_i, k_i) = A(x_{i-1}, k_{i-1})$ and Π_{m+1} is a "brute force proof of unsatisfiability" for the kernel instance then one can prove $\Phi(x, k) \in \overline{SAT}$ by "concatenating" $\Pi_1, \ldots, \Pi_m$ and Π_{m+1}.
- This (usually) yields extended Frege proofs.
- For Frege proofs need $m = O(\log n)$ $(m = O(1))$.

Main (meta)Theorem

- Somewhat too complicated to state precisely.
- If soundness of reduction rules can be witnessed efficiently in Frege, the length of reduction chains is $O(1)$ then unsatisfiable formulas $\Phi(x, k)$ have polynomial size Frege proofs.
- If soundness of reduction rules can be witnessed efficiently in Frege, the length of reduction chains is $O(log(|\Phi(x, k)|))$ then unsatisfiable formulas $\Phi(x, k)$ have quasipolynomial Frege proofs.
- otherwise we normally get polynomial size extended Frege proofs.



- Note: inner cycle already chromatic # 3.
- $A \in \binom{n}{k}$ stable if it doesn't contain consecutive elements i, $i+1$ (including $n, 1$).
- Schrijver's Thm.: Chromatic number of stable Kneser graph is $n - 2k + 2$. A. Schrijver. Vertex-critical Subgraphs of Kneser-graphs. N. Arch. Wiskunde XXVI (1978).

THEOREM: For every $k \geq 1$ Schrijver's theorem has quasi-poly size Frege proofs (poly-size Frege)

- Proof idea: data reduction of length $O(\log n)$.

Critical ingredient

We show that $\Theta(n)$ color classes c are star-shaped, i.e. sets colored with color c have an element in common. Need version of Talbot (Intersecting families of separated sets. Journal of the London Mathematical Society, 68(1):37-51, 2003) that can be simulated propositionally:

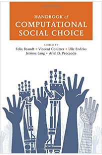
Theorem
If $\mathcal{C}$ is a color class that is not star-shaped then $|\mathcal{C}| \leq k^2 \cdot \binom{n+k-1}{k-2}$.

Thus if there were a $n-2k+1$ coloring c of $SKn_{n,k}$ then we could drop $r = \Theta(n)$ elements of $\{1, 2, \ldots, n\}$ and equally many colors, and reduce the problem to showing that $\chi(SKn_{n-r,k}) > n - r - 2k + 1$.

A Couple of Applications to Proof Complexity

- classical (ad-hoc) kernelization for VertexCover $\Rightarrow$ for every fixed k, negative instances of VC with parameter k have poly-size Frege proofs.
- crown decomposition for DualColoring $\Rightarrow$ negative instances of DualColoring with parameter k poly-size Frege proofs.
- improved (ad-hoc) kernelization for Edge Clique Cover $\Rightarrow$ negative instances (G,k) of Edge Clique Cover have extended Frege proofs of poly size and Frege proofs of quasipoly size.
- sunflower lemma-based kernelization of d-HittingSet $\Rightarrow$ negative instances of d-HittingSet with parameter k extended Frege proofs of poly size.

Applications: Computational Social Choice



- Arrow, Gibbard-Satterthwaite: Fundamental impossibility results on ranking m objects by n agents.
- Tang & Lin (Artificial Intelligence, 2009): Arrow's Theorem has computer-assisted propositional proofs by reducing the general case to the case $n = 2, m = 3$. Similar results (2008) for the Gibbard-Satterthwaite theorem.
- Their proofs: data reductions of length $\Theta(n + m)$.

We give: data reductions of length $O(n)$. Consequently, formulas $Arrow_{m,n}$, $GS_{m,n}$ have (i). quasipoly size Frege proofs (ii). poly size Frege proofs for fixed n.

Conclusions

- Theoretically interesting connections between different areas.
- Work in progress:
  - Adapt this program to other techniques from parameterized complexity, e.g. iterative compression.
  - Adapt this program to other proof systems, e.g. SPR^{-} (Heule,Kiesl & Biere HVC'17, J. Autom. Reasoning '19, Buss & Thapen SAT'19).
  - Proof system that only preserves equisatisfiability, not equivalence
  - Proof complexity for statements in judgment aggregation.
- Proof complexity lower bounds for hard problems in parameterized complexity ?

Where I Would Like to Go

- (Combinatorial) Algebraic Topology: works with exponential size objects.
- Proof Complexity: Cook-Reckow. A proof should be:
  - verifiable in polynomial time.
  - complete.

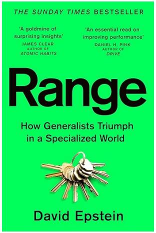
What about non-complete/non-constructive proof systems?

- implicit proofs (Krajicek)
- oracle proof systems (Cook)

On the Combinatorial Algebraic Topology side: e.g. R. Živaljevič. User's guide to equivariant methods in Combinatorics I+II.

On a Personal Level ...

Personal Philosophy:



Modern Science:

- Specialized.
- Compartmentalized.
- Competitive.
- "Megaconferences".

(I hope that I convinced you that) sometimes it pays to straddle multiple scientific topics!

Hvala/Thank You !

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