Canonical and Categorical Rigid Representations of Finite Local-unit-aligned Totally Ordered Monoids

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Finite Totally Ordered Monoids

Throughout the talk,

$$\mathbf{M} = \langle M, \leq, \cdot, e \rangle$$

denotes a **finite totally ordered monoid**. Thus $\langle M, \leq \rangle$ is a finite chain, $\langle M, \cdot, e \rangle$ is a monoid, and multiplication is isotone in each variable; that is, for all $x, y, z \in M$,

$$x \leq y \implies xz \leq yz \quad \text{and} \quad zx \leq zy.$$

Local-unit-aligned Totally Ordered Monoids

For $x \in M$, define

$$R(x) := \{z \in M : xz = x\} \qquad \text{and} \qquad L(x) := \{z \in M : zx = x\}.$$

Since $e \in R(x) \cap L(x)$ and the chain is finite, both sets have greatest elements. We write

$$\tau_r(x) := \max R(x) \qquad \text{and} \qquad \tau_\ell(x) := \max L(x).$$

The monoid M is called **local-unit-aligned** if for every $x \in M$ the two maxima coincide,

$$\tau_r(x) = \tau_\ell(x).$$

In this case we write

$$\tau(x) := \tau_r(x) = \tau_\ell(x)$$

and call τ the **local-unit map** of M.

Commutative totally ordered monoids

Every finite totally ordered commutative monoid is local-unit-aligned, since the left and right local-unit conditions coincide; hence the theory covers the entire commutative case.

Structure and Functoriality

1) The local-unit map τ yields a canonical rigid direct-system representation by **τ-multiplication-cohesive** components.

2) Two complementary morphism theories turn this representation into **equivalences of categories**.

Canonical Direct System Decomposition

Compute τ

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

Figure 1: The multiplication table of the example monoid.

Compute τ

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

29

Figure 2: Computation of τ: the stabilizer set of 1 is highlighted, and its greatest element, $\tau(1)$, is displayed.

Compute τ

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

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Figure 3: Computation of τ: the stabilizer set of 2 is highlighted, and its greatest element, $\tau(2)$, is displayed.

Compute τ

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

29 28 27

Figure 4: Computation of τ: the stabilizer set of 3 is highlighted, and its greatest element, $\tau(3)$, is displayed.

Compute τ

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29	
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29	
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29	
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29	
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29
	29	28	27	28	27	26	24	26	19	16	15	14	15	14	15	16	16	16	19	20	21	21	23	24	25	26	27	28	29

Figure 5: The computation of τ is complete: all τ-values are displayed together with their stabilizer sets.

The range of τ is $\mathrm{Id}^+(M)$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

29	28	27	28	27	26	24	26	19	16	15	14	15	14	15	16	16	16	19	20	21	21	23	24	25	26	27	28	29

Figure 6: The row and column of the unit element 14 are marked in red; the positive idempotents are marked in the 0th column, and their idempotence cells are highlighted.

Find defect edges

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

29	28	27	28	27	26	24	26	19	16	15	14	15	14	15	16	16	16	19	20	21	21	23	24	25	26	27	28	29

Figure 7: Since $3 \cdot 3 = 2$, $\tau(3) = 27$, and $\tau(2) = 28$, we have $\tau(3) \vee \tau(3) = 27 \neq 28 = \tau(3 \cdot 3)$, so $27-28$ is a defect edge.

Find defect edges

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29	
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29	
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29	
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29	
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

29	28	27	28	27	26	24	26	19	16	15	14	15	14	15	16	16	16	19	20	21	21	23	24	25	26	27	28	29

Figure 8: Since $17 \cdot 18 = 19$, $\tau(17) = 16$, and $\tau(18) = 16$, we have $\tau(17) \vee \tau(18) = 16 \neq 19 = \tau(17 \cdot 18)$, so $16-19$ is also a defect edge.

Find defect edges

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
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15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
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21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
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29	28	27	28	27	26	24	26	19	16	15	14	15	14	15	16	16	16	19	20	21	21	23	24	25	26	27	28	29

Figure 9: Since $7 \cdot 25 = 8$, $\tau(7) = 24$, $\tau(25) = 25$, and $\tau(8) = 26$, we have $\tau(7) \vee \tau(25) = 25 \neq 26 = \tau(7 \cdot 25)$, so $25-26$ is a third defect edge.

Connected components

Canonical blocks

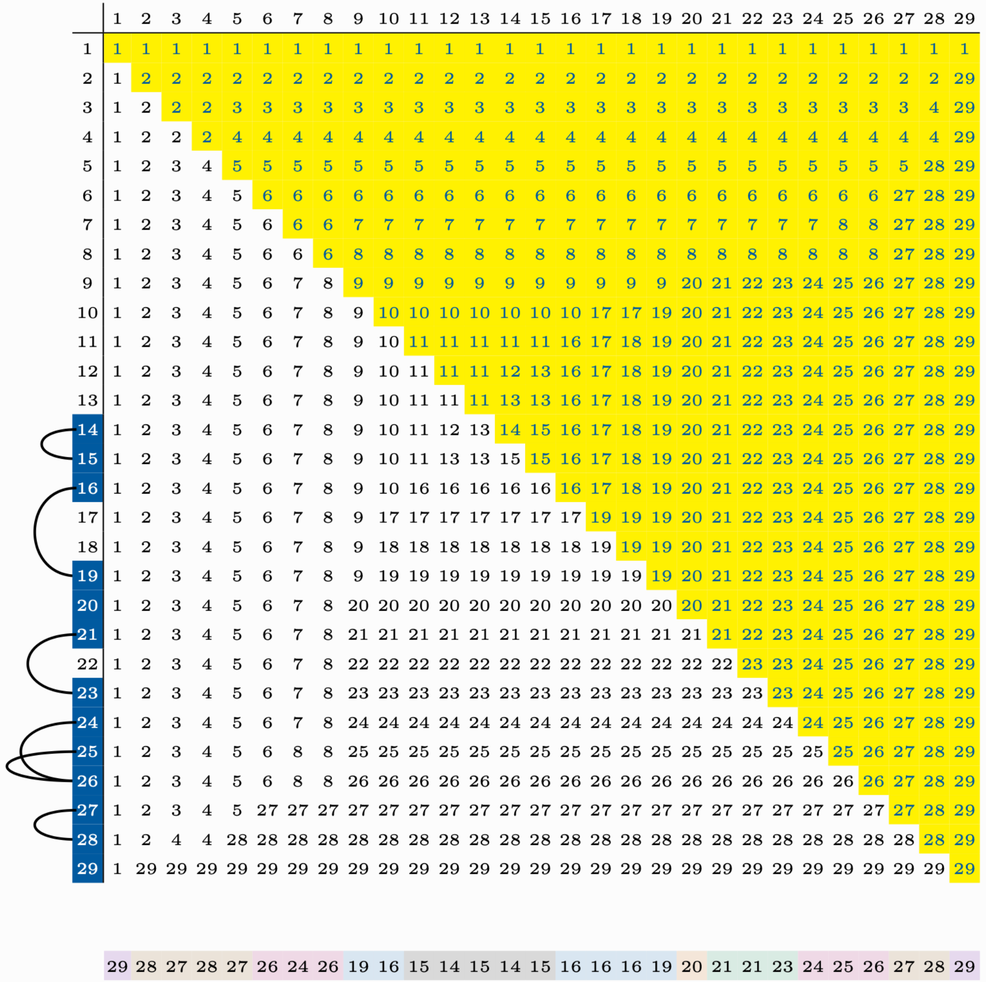

Figure 10: All defect edges are displayed in the 0th column; the connected components of the resulting defect graph determine the block coloring of the positive idempotents and of the τ-row.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
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2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
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10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
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22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
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26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
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28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
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29	28	27	28	27	26	24	26	19	16	15	14	15	14	15	16	16	16	19	20	21	21	23	24	25	26	27	28	29

Figure 11: The canonical τ-multiplication-coherent partition of the positive idempotent skeleton is transferred to the τ-row and then inherited by the carrier, thereby inducing its τ-multiplication-coherent layer decomposition.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
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2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
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9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
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17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
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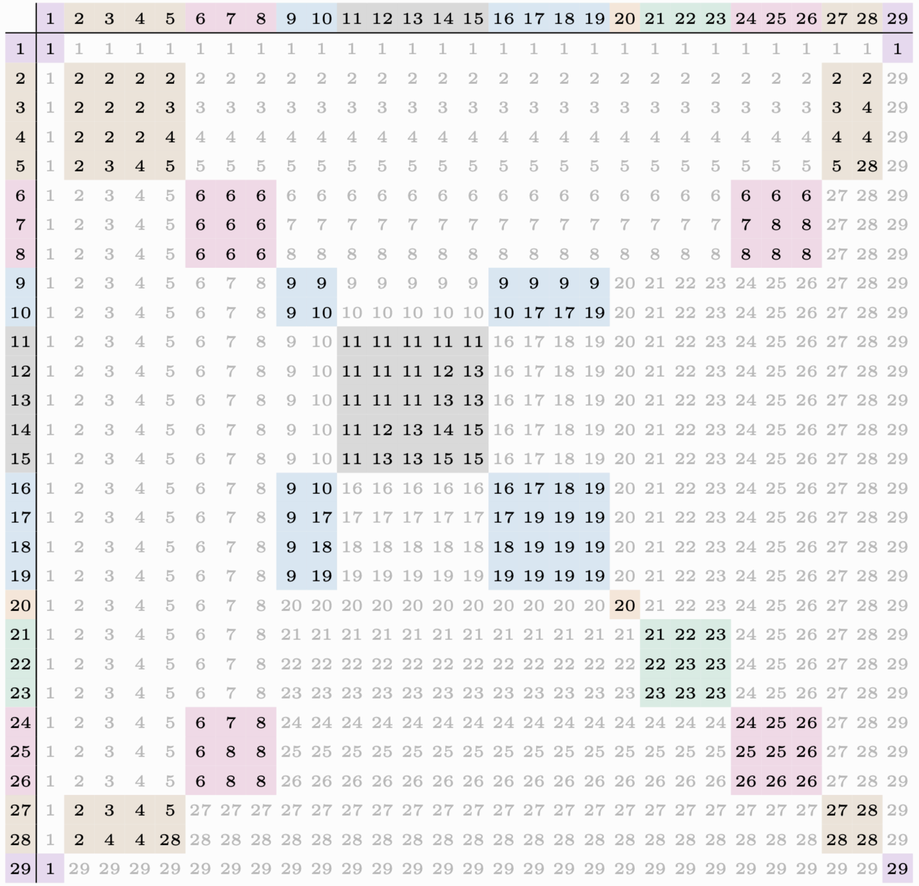
Figure 12: The resulting τ-multiplication-coherent layer decomposition of the carrier.

The canonical τ-multiplication-coherent partition supports a direct-system representation. Moreover, it is the finest partition of M with this property.

Component monoids



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7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
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11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
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16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
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29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

Figure 13: The multiplication internal to each τ-multiplication-coherent layer is highlighted in the ambient Cayley table; each highlighted diagonal block is the Cayley table of the corresponding component monoid.

Component monoids

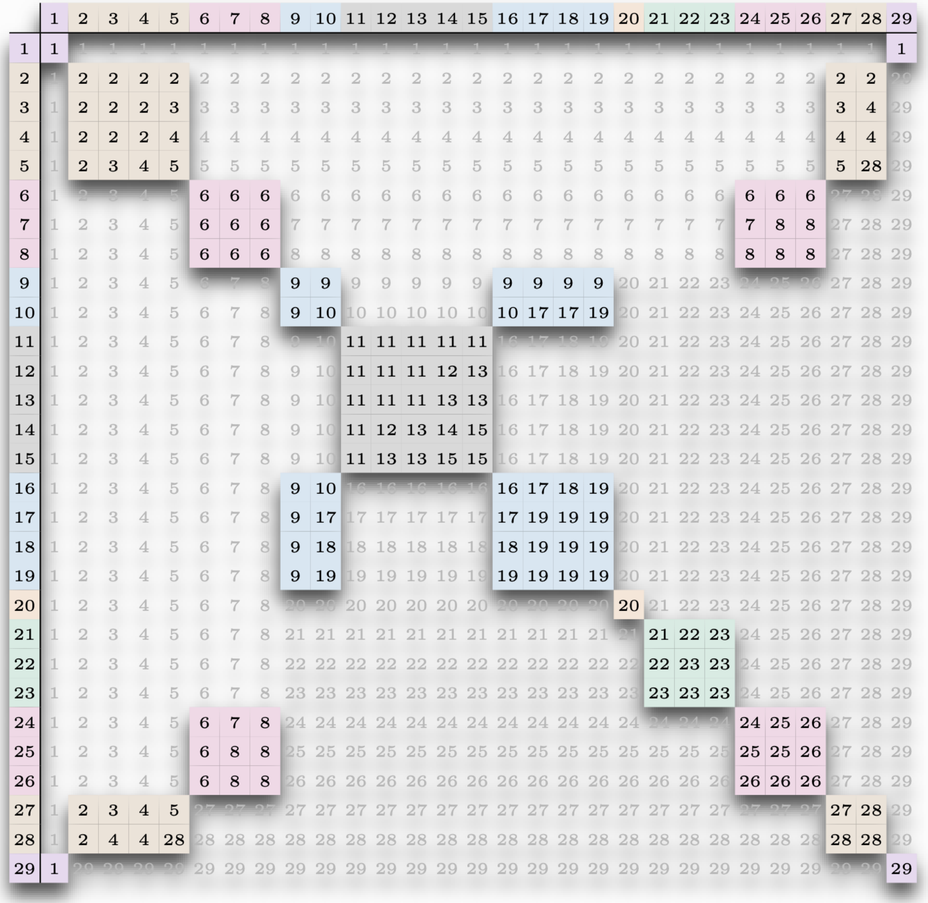


Figure 13: The multiplication internal to each τ-multiplication-coherent layer is highlighted in the ambient Cayley table; each highlighted diagonal block is the Cayley table of the corresponding component monoid.

Component monoids

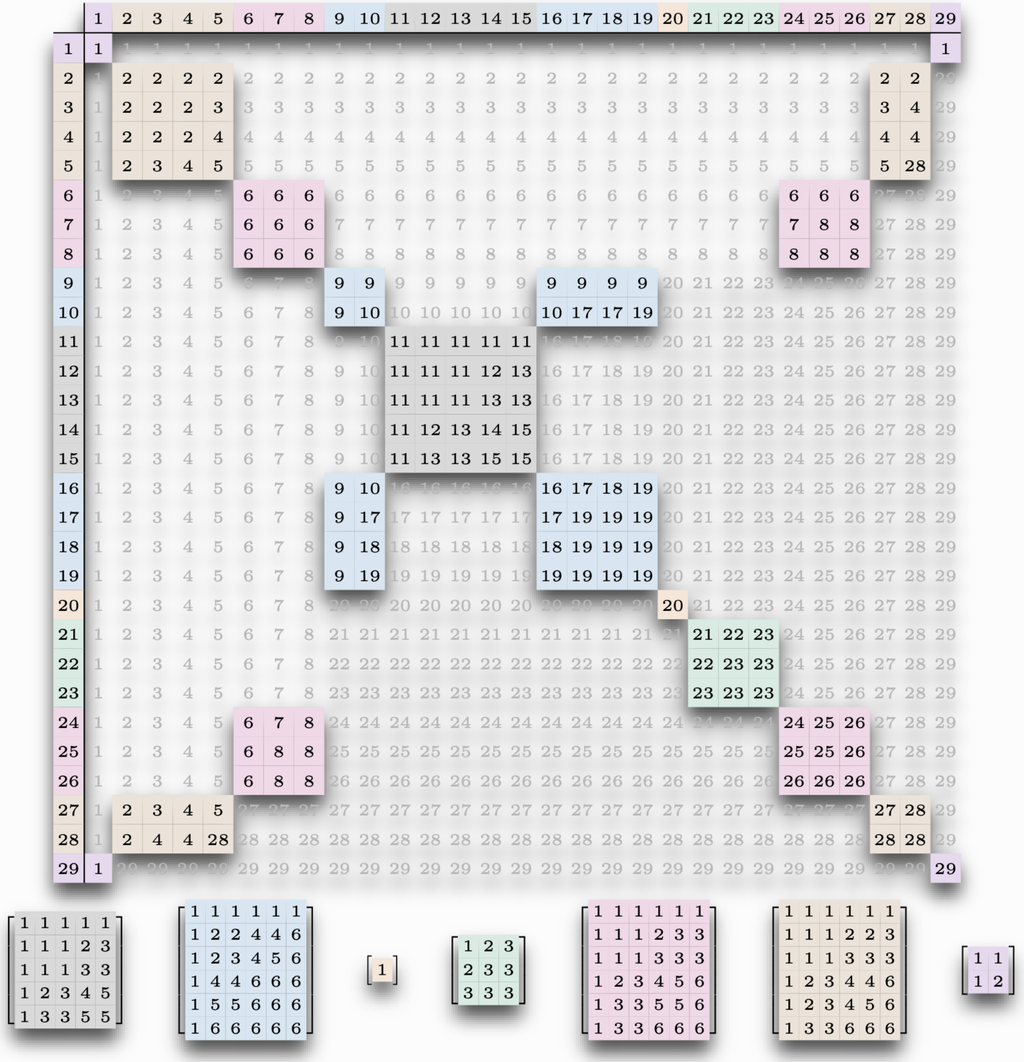


Figure 14: Isomorphic copies of the component monoids, presented on standard carriers of the form $\{1, \ldots, k\}$, are displayed beneath the ambient Cayley table for ease of comparison.

Component monoids

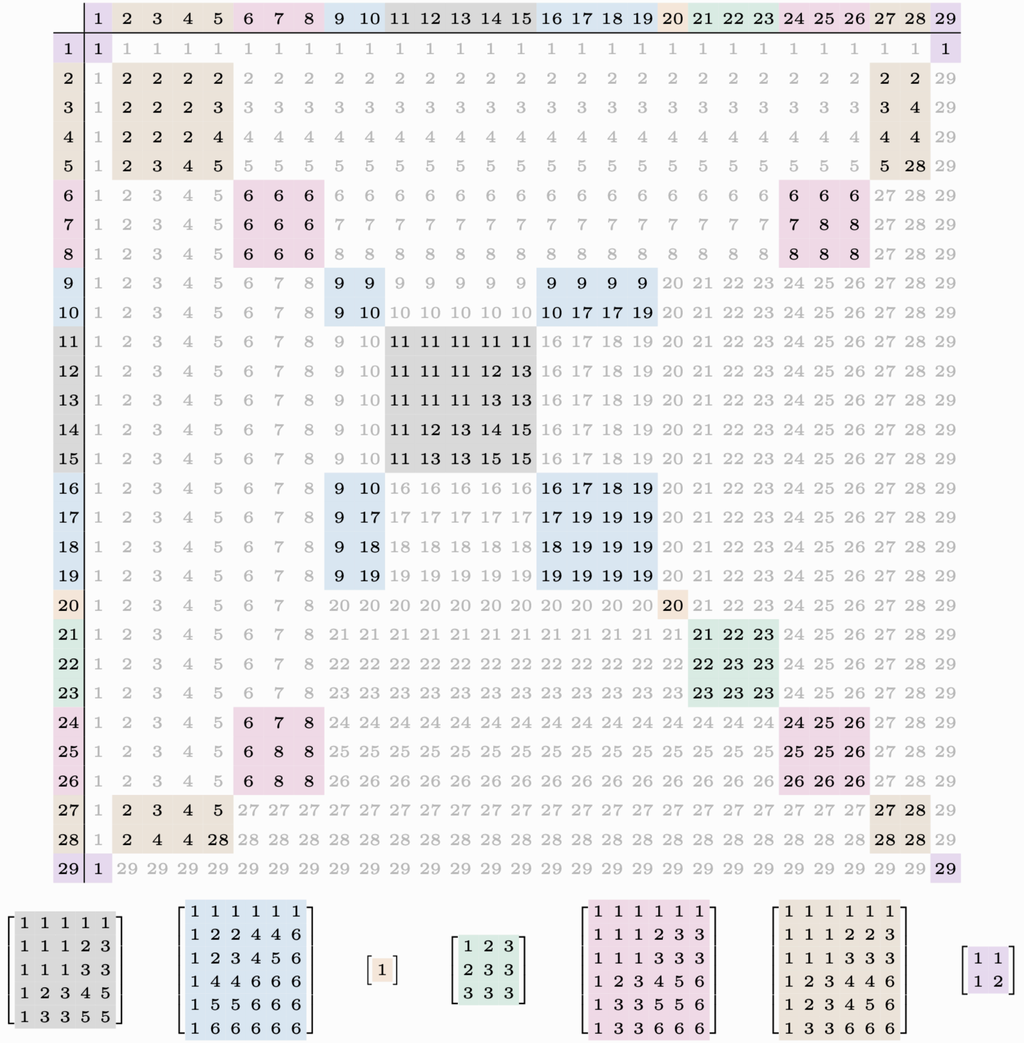
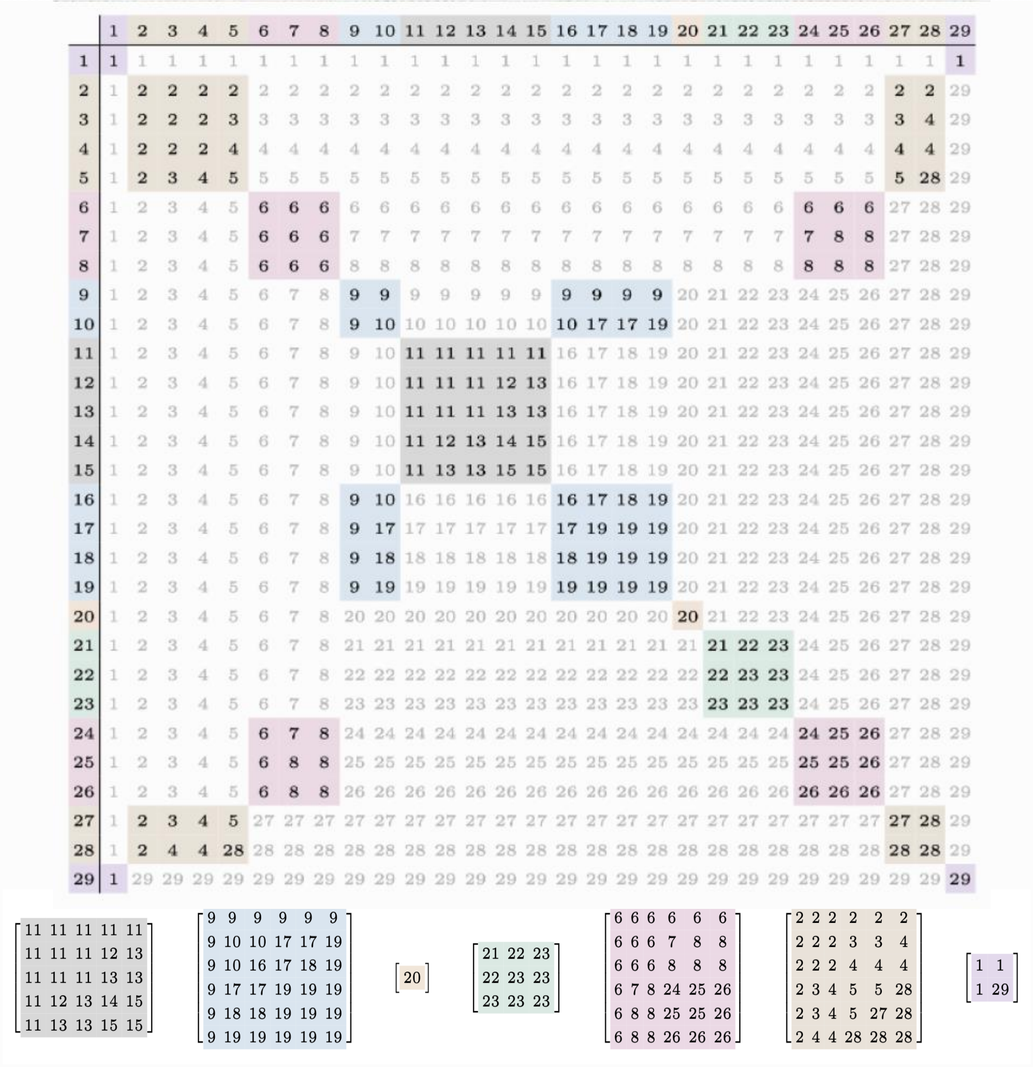


Figure 14: Isomorphic copies of the component monoids, presented on standard carriers of the form $\{1, \ldots, k\}$, are displayed beneath the ambient Cayley table for ease of comparison.

Component monoids



These colored pieces are not merely unital subsemigroups; together they form a canonical direct system, and finiteness makes that system rigid.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	22	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

$$\begin{bmatrix} 11 & 11 & 11 & 11 & 11 \\ 11 & 11 & 11 & 12 & 13 \\ 11 & 11 & 11 & 13 & 13 \\ 11 & 12 & 13 & 14 & 15 \\ 11 & 13 & 13 & 15 & 15 \end{bmatrix} \quad \begin{bmatrix} 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 10 & 10 & 17 & 17 & 19 \\ 9 & 10 & 16 & 17 & 18 & 19 \\ 9 & 17 & 17 & 19 & 19 & 19 \\ 9 & 18 & 18 & 19 & 19 & 19 \\ 9 & 19 & 19 & 19 & 19 & 19 \end{bmatrix} \quad \begin{bmatrix} 20 \end{bmatrix} \quad \begin{bmatrix} 21 & 22 & 23 \\ 22 & 23 & 23 \\ 23 & 23 & 23 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 7 & 8 & 8 \\ 6 & 6 & 6 & 8 & 8 & 8 \\ 6 & 7 & 8 & 24 & 25 & 26 \\ 6 & 8 & 8 & 25 & 25 & 26 \\ 6 & 8 & 8 & 26 & 26 & 26 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 2 & 2 & 4 & 4 & 4 \\ 2 & 3 & 4 & 5 & 5 & 28 \\ 2 & 3 & 4 & 5 & 27 & 28 \\ 2 & 4 & 4 & 28 & 28 & 28 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 29 \end{bmatrix}$$

Direct system

$$A_1 < \cdots < A_k$$

$$M_{A_i} = L_{A_i}, \qquad e_{A_i} = \min A_i$$

$$\rho_{A_i,A_j}(x) = xe_{A_j} \qquad (i \le j).$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
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27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

$$\begin{bmatrix} 11 & 11 & 11 & 11 & 11 \\ 11 & 11 & 11 & 12 & 13 \\ 11 & 11 & 11 & 13 & 13 \\ 11 & 12 & 13 & 14 & 15 \\ 11 & 13 & 13 & 15 & 15 \end{bmatrix} \quad \begin{bmatrix} 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 10 & 10 & 17 & 17 & 19 \\ 9 & 10 & 16 & 17 & 18 & 19 \\ 9 & 17 & 17 & 19 & 19 & 19 \\ 9 & 18 & 18 & 19 & 19 & 19 \\ 9 & 19 & 19 & 19 & 19 & 19 \end{bmatrix} \quad \begin{bmatrix} 20 \end{bmatrix} \quad \begin{bmatrix} 21 & 22 & 23 \\ 22 & 23 & 23 \\ 23 & 23 & 23 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 7 & 8 & 8 \\ 6 & 6 & 6 & 8 & 8 & 8 \\ 6 & 7 & 8 & 24 & 25 & 26 \\ 6 & 8 & 8 & 25 & 25 & 26 \\ 6 & 8 & 8 & 26 & 26 & 26 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 2 & 2 & 4 & 4 & 4 \\ 2 & 3 & 4 & 5 & 5 & 28 \\ 2 & 3 & 4 & 5 & 27 & 28 \\ 2 & 4 & 4 & 28 & 28 & 28 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 29 \end{bmatrix}$$

Rigidity

$$i < j \quad \Longrightarrow \quad \rho_{A_i, A_j}(x) = e_{A_j} \text{ for every } x \in M_{A_i}.$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
16	1	2	3	4	5	6	7	8	9	10	16	16	16	16	16	16	17	18	19	20	21	22	23	24	25	26	27	28	29
17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	22	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
26	1	2	3	4	5	6	8	8	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	27	28	29
27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
28	1	2	4	4	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

$$\begin{bmatrix} 11 & 11 & 11 & 11 & 11 \\ 11 & 11 & 11 & 12 & 13 \\ 11 & 11 & 11 & 13 & 13 \\ 11 & 12 & 13 & 14 & 15 \\ 11 & 13 & 13 & 15 & 15 \end{bmatrix} \quad \begin{bmatrix} 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 10 & 10 & 17 & 17 & 19 \\ 9 & 10 & 16 & 17 & 18 & 19 \\ 9 & 17 & 17 & 19 & 19 & 19 \\ 9 & 18 & 18 & 19 & 19 & 19 \\ 9 & 19 & 19 & 19 & 19 & 19 \end{bmatrix} \quad \begin{bmatrix} 20 \end{bmatrix} \quad \begin{bmatrix} 21 & 22 & 23 \\ 22 & 23 & 23 \\ 23 & 23 & 23 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 7 & 8 & 8 \\ 6 & 6 & 6 & 8 & 8 & 8 \\ 6 & 7 & 8 & 24 & 25 & 26 \\ 6 & 8 & 8 & 25 & 25 & 26 \\ 6 & 8 & 8 & 26 & 26 & 26 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 2 & 2 & 4 & 4 & 4 \\ 2 & 3 & 4 & 5 & 5 & 28 \\ 2 & 3 & 4 & 5 & 27 & 28 \\ 2 & 4 & 4 & 28 & 28 & 28 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 29 \end{bmatrix}$$

Reconstruction.

$$x \in M_A, \quad y \in M_B, \quad C = \max\{A, B\}$$

$$\boxed{xy = \rho_{A,C}(x)\,\rho_{B,C}(y)}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	29
3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
4	1	2	2	2	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	29
5	1	2	3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	28	29
6	1	2	3	4	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	27	28	29
7	1	2	3	4	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	8	8	27	28	29
8	1	2	3	4	5	6	6	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	27	28	29
9	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9	9	9	20	21	22	23	24	25	26	27	28	29
10	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	17	17	19	20	21	22	23	24	25	26	27	28	29
11	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	1	2	3	4	5	6	7	8	9	10	11	11	11	12	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
13	1	2	3	4	5	6	7	8	9	10	11	11	11	13	13	16	17	18	19	20	21	22	23	24	25	26	27	28	29
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
15	1	2	3	4	5	6	7	8	9	10	11	13	13	15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
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17	1	2	3	4	5	6	7	8	9	17	17	17	17	17	17	17	19	19	19	20	21	22	23	24	25	26	27	28	29
18	1	2	3	4	5	6	7	8	9	18	18	18	18	18	18	18	19	19	19	20	21	22	23	24	25	26	27	28	29
19	1	2	3	4	5	6	7	8	9	19	19	19	19	19	19	19	19	19	19	20	21	22	23	24	25	26	27	28	29
20	1	2	3	4	5	6	7	8	20	20	20	20	20	20	20	20	20	20	20	20	21	22	23	24	25	26	27	28	29
21	1	2	3	4	5	6	7	8	21	21	21	21	21	21	21	21	21	21	21	21	21	22	23	24	25	26	27	28	29
22	1	2	3	4	5	6	7	8	22	22	22	22	22	22	22	22	22	22	22	22	22	23	23	24	25	26	27	28	29
23	1	2	3	4	5	6	7	8	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	24	25	26	27	28	29
24	1	2	3	4	5	6	7	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	26	27	28	29
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27	1	2	3	4	5	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	28	29
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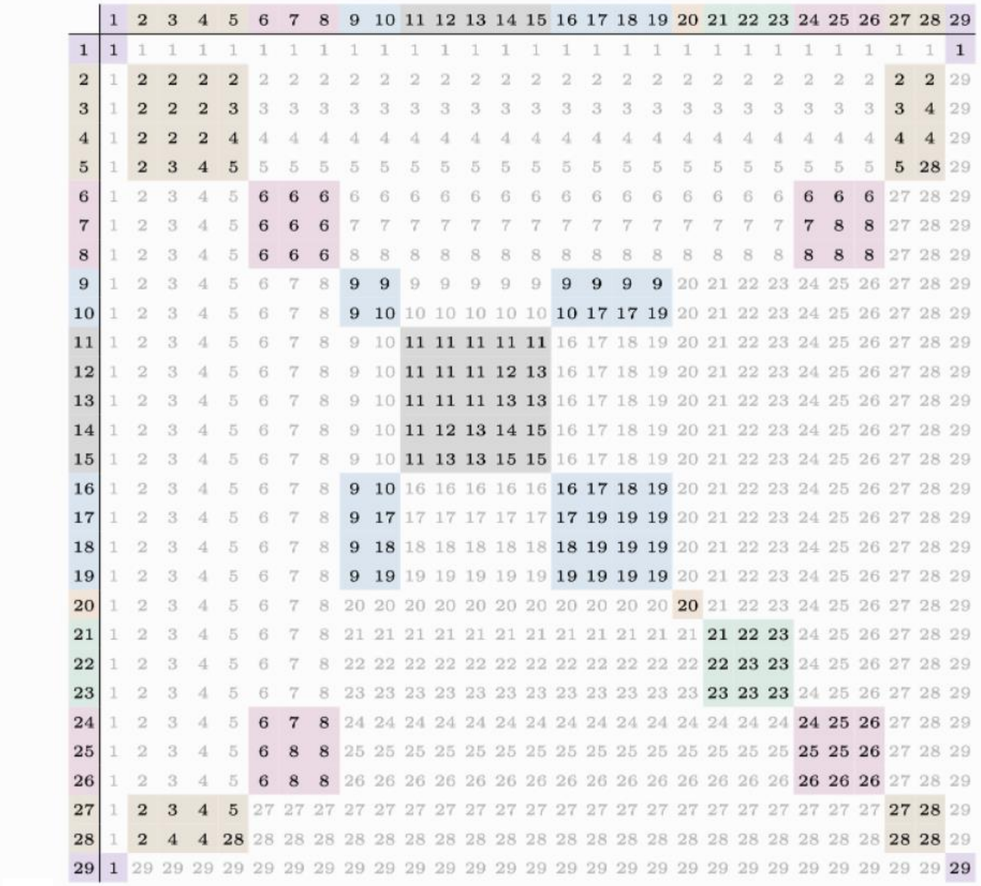
29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

$$\begin{bmatrix} 11 & 11 & 11 & 11 & 11 \\ 11 & 11 & 11 & 12 & 13 \\ 11 & 11 & 11 & 13 & 13 \\ 11 & 12 & 13 & 14 & 15 \\ 11 & 13 & 13 & 15 & 15 \end{bmatrix} \quad \begin{bmatrix} 9 & 9 & 9 & 9 & 9 & 9 \\ 9 & 10 & 10 & 17 & 17 & 19 \\ 9 & 10 & 16 & 17 & 18 & 19 \\ 9 & 17 & 17 & 19 & 19 & 19 \\ 9 & 18 & 18 & 19 & 19 & 19 \\ 9 & 19 & 19 & 19 & 19 & 19 \end{bmatrix} \quad \begin{bmatrix} 20 \end{bmatrix} \quad \begin{bmatrix} 21 & 22 & 23 \\ 22 & 23 & 23 \\ 23 & 23 & 23 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 7 & 8 & 8 \\ 6 & 6 & 6 & 8 & 8 & 8 \\ 6 & 7 & 8 & 24 & 25 & 26 \\ 6 & 8 & 8 & 25 & 25 & 26 \\ 6 & 8 & 8 & 26 & 26 & 26 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 2 & 2 & 4 & 4 & 4 \\ 2 & 3 & 4 & 5 & 5 & 28 \\ 2 & 3 & 4 & 5 & 27 & 28 \\ 2 & 4 & 4 & 28 & 28 & 28 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 29 \end{bmatrix}$$

Reconstruction.

$$A < B \quad \Longrightarrow \quad xy = e_B\, y = y, \qquad yx = y\, e_B = y.$$

collapse to absorption.



$$A < B,\ x \in M_A,\ y \in M_B: \qquad x \le y \iff \rho_{A,B}(x) \le y,$$

Reconstruction.

after rigidity this becomes

$$x \le y \iff e_B \le y.$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
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3	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	29
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25	1	2	3	4	5	6	8	8	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	26	27	28	29
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29	1	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29

$$\begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&2&3 \\ 1&1&1&3&3 \\ 1&2&3&4&5 \\ 1&3&3&5&5 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&1&1 \\ 1&2&2&4&4&6 \\ 1&2&3&4&5&6 \\ 1&4&4&6&6&6 \\ 1&5&5&6&6&6 \\ 1&6&6&6&6&6 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix} \quad \begin{bmatrix} 1&2&3 \\ 2&3&3 \\ 3&3&3 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&1&1 \\ 1&1&1&2&3&3 \\ 1&1&1&3&3&3 \\ 1&2&3&4&5&6 \\ 1&3&3&5&5&6 \\ 1&3&3&6&6&6 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&1&1 \\ 1&1&1&2&2&3 \\ 1&1&1&3&3&3 \\ 1&2&3&4&4&6 \\ 1&2&3&4&5&6 \\ 1&3&3&6&6&6 \end{bmatrix} \quad \begin{bmatrix} 1&1 \\ 1&2 \end{bmatrix}$$

Figure 15: The canonical τ-multiplication-coherent partition of the positive idempotent skeleton is transferred via the τ-row to the carrier, yielding its τ-multiplication-coherent layer decomposition. The multiplication internal to each layer yields a component monoid, displayed below on a standard carrier. The cross-component multiplication is rigid: the product is always the factor belonging to the higher-index component. Equivalently, this factor is the minimum when it lies below its component unit, and the maximum when it lies at or above its component unit. Thus the layer decomposition together with the component monoids determines the ambient multiplication.

Strong semilattice idea + Clifford ordinal sum behaviour unified through finite rigidity

	Clifford: strong semilattices of groups	Clifford: ordered semigroup decomposition	Present representation
Components	Groups	Ordered semigroup components	Local-unit-aligned totally ordered monoids
Indexing structure	Semilattice of idempotents	Totally ordered hierarchy	Finite chain of canonical τ-blocks
Connecting data	Connecting homomorphisms	Cross-level absorption	Canonical transition homomorphisms
Rigidity / absorption	Connecting maps may be nontrivial	Absorption is built into the ordinal-sum structure	Finite rigidity forces every proper transition map to be unit-constant; absorption follows
Setting / assumptions	Inverse semigroups with central idempotents	Naturally totally ordered commutative semigroups	Finite local-unit-aligned totally ordered monoids; no inverses or commutativity assumed

Strong semilattice idea + Clifford ordinal sum behaviour unified through finite rigidity

Unit-constancy (and thus rigidity) is not inherited from Clifford's strong-semilattice theorem: there the connecting homomorphisms may be nontrivial.

Here it is forced by **total order** and **finiteness**, which is precisely what turns the direct system into an ordinal-sum-like structure.

Clifford connections

The present theory therefore has two Clifford-type aspects.

It is Clifford-like in the strong-semilattice sense because it reconstructs a global monoid from components and canonical maps.

It is also Clifford-like in the ordinal-sum sense because, once finiteness forces rigidity, those canonical maps become degenerate enough that the direct-system representation behaves like an ordered component sum with absorption across levels.

Canonical Rigid Direct System Decomposition

What we have just seen in the example holds canonically for every M:

1) The positive idempotents, with the inherited order, form the **positive idempotent skeleton** $\mathrm{Id}^{+}(M)$.

2) Its canonical τ-multiplication-coherent partition induces a direct system decomposition of M: component monoids and transitions.

3) These components are **τ-multiplication-cohesive**: the same construction cannot split them any further.

Back-and-Forth Theorem

Decompose, then reconstruct: for every finite totally ordered local-unit-aligned monoid M, $M \longrightarrow (M_1,\ldots,M_k) \longrightarrow M'$ with $M' \cong M$ canonically, where the M_i are τ-multiplication-cohesive.

Construct, then decompose: starting from any ordered sequence $(M_1,\ldots,M_k)$ of finite totally ordered local-unit-aligned monoids, the canonical decomposition of the resulting monoid may further refine the prescribed components. If all M_i are already τ-multiplication-cohesive, no further refinement occurs: the canonical decomposition recovers the original ordered sequence exactly.

The Functorial Question

Object level. The canonical rigid direct system D(M) records exactly the data needed to reconstruct M.

Morphism level. Which homomorphisms f: M → N correspond exactly to morphisms D(M) → D(N)?

Two natural answers arise from the same rigid objects, but they use different notions of morphism.

Strict Block Morphisms

A strict block morphism preserves the canonical component structure:

$$f(M_A) \subseteq N_{\sigma(A)} \qquad \text{and} \qquad f(e_A) = e_{\sigma(A)}$$

Thus it preserves each component and its distinguished block unit.

Strict Block Morphisms

A strict block morphism induces an isotone map on canonical blocks:

$$\sigma_f : C_m(M) \longrightarrow C_m(N)$$

So the morphism respects the ordered component skeleton.

Strict Block Morphisms

A rigid direct system is a finite chain of τ-multiplication-cohesive component monoids:

$$D = \left((M_i)_{i \in I}, (\rho^D_{i,k})_{i \le k} \right)$$

Every proper transition map is unit-constant.

Strict Block Morphisms

A morphism of rigid systems consists of an isotone index map and component maps:

$$\sigma : I \longrightarrow J, \qquad \Phi_i : M_i \longrightarrow N_{\sigma(i)}$$

Each Φ_i is an isotone unital monoid homomorphism.

Strict Block Morphisms

The component maps are compatible with the transition maps:

$$\Phi_k \circ \rho^D_{i,k} = \rho^E_{\sigma(i),\sigma(k)} \circ \Phi_i$$

Move first inside D, or first via Φ: the two routes agree.

Strict Block Morphisms

A rigid-system morphism gives an ambient homomorphism by applying the component maps piecewise:

$$\mathfrak{M}(\Phi)\big|_{M_i} = \Phi_i$$

Thus morphisms of systems reconstruct morphisms of monoids.

Strict Block Morphisms

Rigid direct systems and their morphisms form a category:

$$\mathbf{RigDS}_{\mathrm{fin}}$$

Strict Block Morphisms

Canonical decomposition and reconstruction are quasi-inverse:

$$\mathbf{LUA}_{\mathrm{fin}}^{\mathrm{blk}} \simeq \mathbf{RigDS}_{\mathrm{fin}}$$

The object-level representation is now functorial.

τ-compatible homomorphisms

A homomorphism is τ-compatible precisely when it preserves local units:

$$\boxed{\tau_N \circ f = f \circ \tau_M}$$

$$\tau_N(f(x)) = f(\tau_M(x)) \qquad (x \in M)$$

τ-compatible homomorphisms

Two immediate consequences of τ-compatibility are:

$$f(u) \in \mathrm{Id}^{+}(N), \qquad f[L_u^M] \subseteq L_{f(u)}^N$$

Thus f respects the intrinsic local-unit geometry, not merely the component structure.

τ-compatible homomorphisms

A τ-compatible homomorphism sends each canonical component into a single canonical component:

$$f[M_A] \subseteq N_{\sigma_f(A)}, \qquad \sigma_f : C_m(M) \longrightarrow C_m(N)$$

The induced block map σ_f is isotone.

τ-compatible homomorphisms

Restrict f to a canonical component:

$$f_A = f|_{M_A} : M_A \longrightarrow N_{\sigma_f(A)}$$

These component maps are isotone, multiplicative and τ-compatible — but they need not be unital.

$f(e_A) \in \mathrm{Id}^+(N_{\sigma_f(A)})$ may differ from $e_{\sigma_f(A)}$

τ-compatible homomorphisms

Let the following denote the category of finite local-unit-aligned totally ordered monoids with τ-compatible homomorphisms:

$$\mathbf{LUA}_{\mathrm{fin}}^{\tau c}$$

τ-compatible homomorphisms

On the direct-system side, the corresponding notion is a collapse morphism:

$$\Phi = \left(\sigma, (\Phi_i)_{i \in I}\right)$$

Its component maps may be non-unital, but its ambient realization is τ-compatible.

τ-compatible homomorphisms

Internal axioms for collapse morphisms

Data $\Phi=(\sigma,(\Phi_i))$ between rigid systems define a τ-compatible collapse morphism iff:

(U) **Global unit**

$$\sigma(i_0) = j_0, \qquad \Phi_{i_0}(e_{i_0}) = e_{j_0}.$$

(C) **Component maps**

Isotone, multiplicative, τ-compatible; not necessarily unital.

$$\tau_{\mathbf{N}_{\sigma(i)}}(\Phi_i(x)) = \Phi_i(\tau_{\mathbf{M}_i}(x)).$$

(F) **Forward cross-level order**

For $i \le k$ and $x \in M_i$:

$$\rho^{\mathbf{E}}_{\sigma(i),\,\sigma(k)}(\Phi_i(x)) \le_{\sigma(k)} \Phi_k(\rho^{\mathbf{D}}_{i,k}(x)).$$

(R) **Reversed cross-level order**

If $i<k$, $x \in M_k$, $y \in M_i$ and $x <_k \rho^{D}_{i,k}(y)$, then:

$$\Phi_k(x) \le_{\sigma(i)} \Phi_i(y) \quad (\sigma(i) = \sigma(k))$$
$$\Phi_k(x) <_{\sigma(k)} \rho^{\mathbf{E}}_{\sigma(i),\,\sigma(k)}(\Phi_i(y)) \quad (\sigma(i) < \sigma(k)).$$

(A) **Collapse absorption**

If $i<k$ and $\sigma(i)=\sigma(k)$, then:

$$\Phi_i(x)\Phi_k(y) = \Phi_k(y) = \Phi_k(y)\Phi_i(x).$$

Thus the ambient condition has a fully internal direct-system form.

τ-compatible homomorphisms

This yields a second categorical equivalence:

$$\mathbf{LUA}_{\text{fin}}^{\tau c} \simeq \mathbf{RigDS}_{\text{fin}}^{\tau c}$$

The same rigid objects support a complementary morphism theory.

Complementary Morphism Theories

	Strict block	τ-compatible
Defining condition	$f(M_A) \subseteq N_{\sigma(A)}$ $f(e_A) = e_{\sigma(A)}$	$\tau_N f = f \tau_M$
Component maps	Isotone unital monoid homomorphisms	Isotone, multiplicative and τ-compatible They may be non-unital
Index collapse	σ is isotone Non-injective maps are allowed	σ is isotone Positive idempotents and components may collapse
Structure remembered	Canonical components and their block units	Local-unit geometry intrinsic to τ

Neither morphism class contains the other.

Complementary Morphism Theories

A **strict block morphism** preserves the canonical decomposition explicitly. For every canonical block A,

$$f(M_A) \subseteq N_{\sigma_f(A)} \qquad \text{and} \qquad f(e_A) = e_{\sigma_f(A)}.$$

Thus it must preserve the **distinguished unit of each component**. However, it is not required to preserve the local-unit map τ pointwise.

Complementary Morphism Theories

A **τ-compatible homomorphism**, by contrast, satisfies the intrinsic equation

$$\tau_M(f(x)) = f(\tau_M(x)) \quad (x \in M).$$

So it preserves the **exact local-unit geometry**. This forces a source canonical component to land inside a single target component, but it only implies

$$f(e_A) \in \sigma_f(A) = \mathrm{Id}^+(N_{\sigma f(A)}),$$

not

$$f(e_A) = e_{\sigma_f(A)}.$$

In other words, the source block unit may be sent to a **nonminimal positive idempotent** of the target block.

Sources

Object-level representation

S. Jenei, A canonical rigid direct-system representation of finite local-unit-aligned totally ordered monoids, **Semigroup Forum** (2026).

DOI: 10.1007/s00233-026-10672-0 arXiv: 2607.23801

Categorical sequel

S. Jenei, Categorical forms of the rigid direct-system representation for finite local-unit-aligned totally ordered monoids. Submitted. arXiv: 2609.20233

Take-Home Message

Finite totally ordered local-unit-aligned monoids are canonically assembled from τ-multiplication-cohesive components.

Rigidity makes the representation functorial for strict block morphisms and for τ-compatible homomorphisms.

Take-Home Message

Finite totally ordered local-unit-aligned monoids are canonically assembled from τ-multiplication-cohesive components.

Rigidity makes the representation functorial for strict block morphisms and for τ-compatible homomorphisms.

Thank you.