

Proof Theoretic Applications of CERES

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LAP 2026 Dubrovnik
September 25, 2026

Motivation

mathematical theorems: used in a variety of areas

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an example: a psychologist does not need the central limit theorem
for using normal distribution

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Are the proofs of theorems important?

	2	3	5	7	11	13	17	19	23
29	31	37	41	43	47	53	59	61	67
71	73	79	83	89	97	101	103	107	109
113	127	131	137	139	149	151	157	163	167
173	179	181	191	193	197	199	211	223	227
229	233	239	241	251	257	263	269	271	277
281	283	293	307	311	313	317	331	337	347
349	353	359	367	373	379	383	389	397	401
409	419	421	431	433	439	443	449	457	461
463	467	479	487	491	499	503	509	521	523
541	547	557	563	569	571	577	587	593	599
601	607	613	617	619	631	641	643	647	653
659	661	673	677	683	691	701	709	719	727
733	739	743	751	757	761	769	773	787	797
809	811	821	823	827	829	839	853	857	859
863	877	881	883	887	907	911	919	929	937
941	947	953	967	971	977	983	991	997	

The topic of this talk...

... is not Euclid's proof. But,

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Euclid proved that there are infinitely many prime numbers.

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Is 27897 a prime?

The topic of this talk...

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Euclid proved that there are infinitely many prime numbers.

Is 27897 a prime? **Euclid's proof can't tell.**

The topic of this talk...

What more do we know if we have proved a theorem than if we merely know that it is true?

proof mining:

extract **additional mathematical information** from proofs

Cut-elimination

Mathematical proofs are based on the structuring of reasoning by intermediate statements (**lemmas**).

$$\frac{\Gamma \vdash A \quad A, \Pi \vdash \Delta}{\Gamma, \Pi \vdash \Delta} \textit{cut}$$

drawback:

- ▶ only the **truth** of a lemma is reflected in the derivation of their statement
- ▶ the proof of a lemma may contain important mathematical information which is lost

eliminating cuts: we get purely combinatorial proofs

Cut-elimination

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eliminating cuts: we get purely combinatorial proofs

cut-elimination theorem

If a statement is provable, then it is provable without cuts.

Automated proof analysis



Automated proof analysis



↓ (automated proof analysis)

Automated proof analysis



⇓ (automated proof analysis)

explicit information:

- ▶ upper and lower bounds on variables in a theorem
- ▶ programs from proofs
- ▶ Herbrand sequents: instantiations of the quantified variables in a theorem
- ▶ interpolants

Cut-elimination by resolution

CERES: Cut-**E**limination by **RE**solution.

Cut-elimination by resolution

CERES: Cut-Elimination by RESolution.

1. Extraction of the characteristic formula F , representing the derivations going to the cuts in φ .

Cut-elimination by resolution

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2. Construction of a refutation γ of F (any theorem prover).

Cut-elimination by resolution

CERES: Cut-**E**limination by **RES**olution.

1. Extraction of the characteristic formula F , representing the derivations going to the cuts in φ .
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3. Construction of a proof projection φ^* of φ .

Cut-elimination by resolution

CERES: Cut-**E**limination by **RES**olution.

1. Extraction of the characteristic formula F , representing the derivations going to the cuts in φ .
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3. Construction of a proof projection φ^* of φ .
4. Combine γ with instances of φ^* to a **CERES normal form** of φ , which contains only quantifier-free cuts.

Cut-elimination by resolution

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3. Construction of a proof projection φ^* of φ .
4. Combine γ with instances of φ^* to a **CERES normal form** of φ , which contains only quantifier-free cuts.
5. **Extract Herbrand substitutions from the CERES normal form.**

M. Baaz, A. Leitsch: Cut-elimination and Redundancy-elimination by Resolution. J. Symb. Comput. 29(2): 149-177 (2000)

CERES normal form

$$\begin{array}{c}
 \frac{
 \frac{
 \frac{
 \frac{
 \frac{
 \Gamma \vdash \Delta, F\theta_1 \quad F\theta_1, \dots, F\theta_n \vdash \quad \text{cut}
 }{
 \Gamma, F\theta_2, \dots, F\theta_n \vdash \Delta
 }
 }{
 \Gamma, F\theta_3, \dots, F\theta_n \vdash \Delta
 }
 }{
 \vdots
 }
 }{
 \Gamma \vdash \Delta, F\theta_n
 }
 }{
 \Gamma \vdash \Delta
 }
 }
 \end{array}$$

$(\varphi^*\theta_1)$ (γ)
 $(\varphi^*\theta_2)$ cut
 $\text{cut} + c^*$

F characteristic formula

γ refutation of F

φ^* projection

CERES normal form

$$\frac{
 \frac{
 \frac{
 \frac{
 \frac{
 \Gamma \vdash \Delta, F\theta_1 \quad F\theta_1, \dots, F\theta_n \vdash \quad \text{cut}
 }{
 \Gamma, F\theta_2, \dots, F\theta_n \vdash \Delta
 }
 }{
 \Gamma, F\theta_3, \dots, F\theta_n \vdash \Delta
 }
 }{
 \vdots
 }
 }{
 \Gamma \vdash \Delta, F\theta_n
 }
 }{
 \Gamma \vdash \Delta
 }
 }$$

$(\varphi^*\theta_1)$ (γ)
 $(\varphi^*\theta_2)$ $(\varphi^*\theta_n)$

F characteristic formula

γ refutation of F

φ^* projection

M. Baaz, A. Leitsch: CERES in Many-Valued Logics. LPAR 2004: 1-20

CERES normal form

$$\frac{\frac{\frac{(\varphi^* \theta_1) \quad (\gamma)}{\Gamma \vdash \Delta, F\theta_1 \quad F\theta_1, \dots, F\theta_n \vdash \Delta} cut}{\Gamma \vdash \Delta, F\theta_2 \quad \Gamma, F\theta_2, \dots, F\theta_n \vdash \Delta} cut}{\Gamma, F\theta_3, \dots, F\theta_n \vdash \Delta} cut + c^*}{\frac{(\varphi^* \theta_n) \quad \vdots}{\Gamma \vdash \Delta, F\theta_n} cut + c^*} \Gamma \vdash \Delta$$

F characteristic formula

 γ refutation of F

φ^* projection

M. Baaz, A. Leitsch: CERES in Many-Valued Logics. LPAR 2004: 1-20

S. Hetzl et al.: CERES in higher-order logic. *Ann. Pure Appl. Log.* 162(12): 1001-1034 (2011)

CERES normal form

$$\frac{\frac{\frac{(\varphi^* \theta_1) \quad (\gamma)}{\Gamma \vdash \Delta, F\theta_1 \quad F\theta_1, \dots, F\theta_n \vdash \Delta} cut}{\Gamma \vdash \Delta, F\theta_2 \quad \Gamma, F\theta_2, \dots, F\theta_n \vdash \Delta} cut}{\Gamma, F\theta_3, \dots, F\theta_n \vdash \Delta} cut + c^*}{\frac{(\varphi^* \theta_n) \quad \vdots}{\Gamma \vdash \Delta, F\theta_n} cut + c^*}{\Gamma \vdash \Delta}$$

F characteristic formula

 γ refutation of F

φ^* projection

M. Baaz, A. Leitsch: CERES in Many-Valued Logics. LPAR 2004: 1-20

S. Hetzl et al.: CERES in higher-order logic. *Ann. Pure Appl. Log.* 162(12): 1001-1034 (2011)

D. M. Cerna et al.: CERES in intuitionistic logic. *Ann. Pure Appl. Log.* 168(10): 1783-1836 (2017)

An application

Proofs of theorems in number theory may use topological structures. Cut-elimination yields proofs without topology.

- ▶ Cut-elimination in [Fürstenberg's proof](#) of the infinitude of primes.
- ▶ Semi-automated analysis via CERES.
- ▶ Cut-elimination CERES gave Euclid's proof.

M. Baaz et al.: CERES: An analysis of Fürstenberg's proof of the infinity of primes. Theor. Comput. Sci. 403(2-3): 160-175 (2008)

Herbrand sequent extraction

$$\begin{array}{c}
 \frac{P(a) \vdash P(a)}{P(a) \vdash P(a), Q(a)} w : r \\
 \frac{P(a) \vdash P(a), Q(a)}{\vdash P(a) \rightarrow Q(a), P(a)} \rightarrow : r \\
 \frac{\vdash \exists y(P(a) \rightarrow Q(y)), P(a)}{B \vdash C, P(a)} \exists : r \\
 \frac{B \vdash C, P(a)}{B, B \vdash C, C, Q(a)} w : l \\
 \frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow : l \\
 \frac{P(a), P(a) \rightarrow Q(a) \vdash Q(a)}{P(a), \forall x(P(x) \rightarrow Q(x)) \vdash Q(a)} \forall : l \\
 \frac{P(a), \forall x(P(x) \rightarrow Q(x)) \vdash Q(a)}{P(a), B \vdash C, Q(a)} w : r \\
 \frac{B, B \vdash C, C, Q(a)}{B, B \vdash C, C, Q(a)} cut \\
 (\pi_1)
 \end{array}$$

$$\begin{array}{c}
 \frac{Q(a) \vdash Q(a)}{P(a), Q(a) \vdash Q(a)} w : l \\
 \frac{P(a), Q(a) \vdash Q(a)}{Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow : r \\
 \frac{Q(a) \vdash P(a) \rightarrow Q(a)}{Q(a) \vdash \exists y P(a) \rightarrow Q(y)} \exists : r \\
 \frac{Q(a) \vdash \exists y P(a) \rightarrow Q(y)}{Q(a), B \vdash C} w : l \\
 \frac{B, B \vdash C, C, Q(a) \quad Q(a), B \vdash C}{B, B, B \vdash C, C, C} cut \\
 \frac{B, B, B \vdash C, C, C}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} c^*
 \end{array}$$

where $B = \forall x(P(x) \rightarrow Q(x))$ and $C = \exists y(P(a) \rightarrow Q(y))$.

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 \frac{P(a) \vdash P(a)}{P(a) \vdash P(a), Q(a)} w : r \\
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 \frac{}{B \vdash C, P(a)} w : l \\
 \hline
 B, B \vdash C, C, Q(a) \\
 (\pi_1)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow : l \\
 \frac{}{P(a), \forall x(P(x) \rightarrow Q(x)) \vdash Q(a)} \forall : l \\
 \frac{}{P(a), B \vdash C, Q(a)} w : r \\
 \hline
 P(a), B \vdash C, Q(a) \quad cut
 \end{array}$$

$$\begin{array}{c}
 \frac{Q(a) \vdash Q(a)}{P(a), Q(a) \vdash Q(a)} w : l \\
 \frac{}{Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow : r \\
 \frac{}{Q(a) \vdash \exists y P(a) \rightarrow Q(y)} \exists : r \\
 \frac{}{Q(a), B \vdash C} w : l \\
 \hline
 B, B \vdash C, C, Q(a) \quad Q(a), B \vdash C \quad cut \\
 \hline
 B, B, B \vdash C, C, C \\
 \hline
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y)) \quad c^*
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 \frac{}{Q(a), B \vdash C} w : l \\
 \hline
 B, B \vdash C, C, Q(a) \quad (\pi_1) \\
 \hline
 B, B, B \vdash C, C, C \quad cut \\
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 \frac{B \vdash C, P(a)}{B, B \vdash \textcolor{red}{C}, C, Q(a)} w : l \\
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 \frac{P(a), \forall x(P(x) \rightarrow Q(x)) \vdash Q(a)}{P(a), B \vdash C, Q(a)} w : r \\
 \frac{B, B \vdash \textcolor{red}{C}, C, Q(a)}{B, B \vdash \textcolor{red}{C}, C, Q(a)} cut \\
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 \frac{Q(a) \vdash \exists y P(a) \rightarrow Q(y)}{Q(a), B \vdash C} w : l \\
 \frac{B, B \vdash \textcolor{red}{C}, C, Q(a)}{B, B \vdash \textcolor{red}{C}, C, C} cut \\
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 \frac{B \vdash \textcolor{red}{C}, P(a)}{B, B \vdash \textcolor{red}{C}, C, Q(a)} w : l \\
 \frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow : l \\
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 \frac{P(a), \forall x(P(x) \rightarrow Q(x)) \vdash Q(a)}{P(a), B \vdash C, Q(a)} w : r \\
 \frac{B, B \vdash \textcolor{red}{C}, C, Q(a) \quad P(a), B \vdash C, Q(a)}{B, B \vdash \textcolor{red}{C}, C, Q(a)} cut \\
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 \frac{B, B \vdash \textcolor{red}{C}, C, Q(a) \quad Q(a), B \vdash C}{B, B, B \vdash \textcolor{red}{C}, C, C} cut \\
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 B, B \vdash \textcolor{red}{C}, C, Q(a) \quad \text{cut} \\
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 B, B, B \vdash \textcolor{red}{C}, C, C \quad c^* \\
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 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))
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 \frac{B \vdash C, P(a)}{B, B \vdash C, C, Q(a)} cut \\
 (\pi_1)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow : l \\
 \frac{P(a), P(a) \rightarrow Q(a) \vdash Q(a)}{P(a), \forall x(P(x) \rightarrow Q(x)) \vdash Q(a)} \forall : l \\
 \frac{P(a), \forall x(P(x) \rightarrow Q(x)) \vdash Q(a)}{P(a), B \vdash C, Q(a)} w : r \\
 \frac{P(a), B \vdash C, Q(a)}{B, B \vdash C, C, Q(a)} cut
 \end{array}$$

$$\begin{array}{c}
 \frac{Q(a) \vdash Q(a)}{P(a), Q(a) \vdash Q(a)} w : l \\
 \frac{P(a), Q(a) \vdash Q(a)}{Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow : r \\
 \frac{Q(a) \vdash P(a) \rightarrow Q(a)}{Q(a) \vdash \exists y P(a) \rightarrow Q(y)} \exists : r \\
 \frac{Q(a) \vdash \exists y P(a) \rightarrow Q(y)}{Q(a), B \vdash C} w : l \\
 \frac{Q(a), B \vdash C}{B, B \vdash C, C, Q(a)} cut \\
 \frac{B, B \vdash C, C, Q(a)}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} c^*
 \end{array}$$

where $B = \forall x(P(x) \rightarrow Q(x))$ and $C = \exists y(P(a) \rightarrow Q(y))$.

CERES normal form

$$\begin{array}{c}
 \begin{array}{c}
 (\varphi^*\theta_2) \qquad \qquad \qquad (\varphi^*\theta_1) \qquad \qquad \qquad (\gamma) \\
 \Gamma \vdash \Delta, F\theta_2 \qquad \frac{\Gamma \vdash \Delta, F\theta_1 \qquad F\theta_1, \dots, F\theta_n \vdash}{\Gamma, F\theta_2, \dots, F\theta_n \vdash \Delta} cut \\
 \hline
 \Gamma, F\theta_3, \dots, F\theta_n \vdash \Delta \qquad cut + c^*
 \end{array} \\
 \vdots \\
 \frac{\begin{array}{c} (\varphi^*\theta_n) \\ \Gamma \vdash \Delta, F\theta_n \end{array} \qquad \qquad \qquad \Gamma \vdash \Delta}{\Gamma \vdash \Delta} cut + c^*
 \end{array}$$

CERES normal form

$$\begin{array}{c}
 \frac{\frac{\frac{(\varphi^* \theta_2)}{\Gamma \vdash \Delta, F\theta_2} \quad \frac{\frac{(\varphi^* \theta_1)}{\Gamma \vdash \Delta, F\theta_1} \quad \frac{F\theta_1, \dots, F\theta_n \vdash}{\Gamma, F\theta_2, \dots, F\theta_n \vdash \Delta} \text{cut}}{\Gamma, F\theta_3, \dots, F\theta_n \vdash \Delta} \text{cut} + c^*}{\Gamma \vdash \Delta, F\theta_n} \quad \vdots}{\Gamma \vdash \Delta} \text{cut} + c^*
 \end{array}$$

(γ)

The proof projection φ^* contains the **substitutions for quantified formulas in the end-sequent**, it can be considered as the *logical* part of the CERES normal form.

But: we have to take $\theta_1, \dots, \theta_n$ into account!

The CERES method (revised): Herbrand extraction

1. Extraction of the characteristic formula F , representing the derivations going into the cuts of φ .
2. Construction of a proof projection φ^* of φ . **Extract Herbrand substitutions from φ^* .**
3. Construction of a refutation γ of F (any theorem prover). **Extract Herbrand substitutions from γ .**
4. Combine the Herbrand substitutions of φ^* and γ to compute the Herbrand sequent of a CERES normal form.

Example

Let φ be the proof (for u, v free variables, a a constant symbol)

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad \frac{Q(u) \vdash Q(u)}{P(u), \neg P(u) \vee Q(u) \vdash Q(u)} \vee : I \\
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad \frac{Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \vee : r \\
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : r \\
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash (\exists y) Q(y)} \forall : I \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y) \quad \text{cut}
 \end{array}$$

Example: the characteristic formula

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{P(u), \neg P(u) \vee Q(u) \vdash Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \hline
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad Q(v) \vdash Q(v) \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \exists : r \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : I \\
 \hline
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)} \forall : I \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y) \quad \text{cut}
 \end{array}$$

Example: the characteristic formula

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{P(u), \neg P(u) \vee Q(u) \vdash Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \hline
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad Q(v) \vdash Q(v) \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \exists : r \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : I \\
 \hline
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)} \forall : I \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y) \quad \text{cut}
 \end{array}$$

$$(\neg P(u) \vee Q(u))$$

Example: the characteristic formula

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{P(u), \neg P(u) \vee Q(u) \vdash Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \hline
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad Q(v) \vdash Q(v) \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \exists : r \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : I \\
 \hline
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)} \forall : I \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y) \quad \text{cut}
 \end{array}$$

$$(\neg P(u) \vee Q(u))$$

$$P(a) \wedge \neg Q(v)$$

Example: the characteristic formula

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad \frac{Q(u) \vdash Q(u)}{\neg P(u), \neg P(u) \vee Q(u) \vdash Q(u)} \vee : I \\
 \frac{P(u), \neg P(u) \vee Q(u) \vdash Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \neg : r \\
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \vee : r \\
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \forall : I \\
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : r \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad \frac{Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \exists : r \\
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : I \\
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)} \forall : I \\
 \hline
 \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y) \quad \text{cut}
 \end{array}$$

$$(\neg P(u) \vee Q(u)) \quad P(a) \wedge \neg Q(v)$$

$$F = (\neg P(u) \vee Q(u)) \wedge P(a) \wedge \neg Q(v).$$

Example: the proof projection

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{P(u), \neg P(u) \vee Q(u) \vdash Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} \forall : r
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \hline
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad Q(v) \vdash Q(v) \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \exists : r \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : I \\
 \hline
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)} \forall : I \\
 \hline
 \frac{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} cut
 \end{array}$$

$\Downarrow$

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\vdash P(u), \neg P(u)} \neg : r \\
 \hline
 \frac{\vdash P(u), \neg P(u)}{\neg P(u) \vdash \neg P(u)} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{\neg P(u) \vdash \neg P(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\forall x(\neg P(x) \vee Q(x)) \vdash \neg P(u) \vee Q(u)} \forall : I \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \neg P(u) \vee Q(u)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y), (\neg P(u) \vee Q(u)) \wedge (P(a) \wedge \neg Q(v))} \forall : I
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \quad \frac{Q(v) \vdash Q(v)}{\vdash Q(v), \neg Q(v)} \neg : r \\
 \hline
 \frac{\forall x P(x) \vdash P(a) \quad \vdash Q(v), \neg Q(v)}{\forall x P(x) \vdash Q(v), P(a) \wedge \neg Q(v)} \wedge : r \\
 \hline
 \frac{\forall x P(x) \vdash Q(v), P(a) \wedge \neg Q(v)}{\forall x P(x) \vdash \exists y Q(y), P(a) \wedge \neg Q(v)} \exists : r \\
 \hline
 \frac{\forall x P(x) \vdash \exists y Q(y), P(a) \wedge \neg Q(v)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y), (\neg P(u) \vee Q(u)) \wedge (P(a) \wedge \neg Q(v))} \wedge : r
 \end{array}$$

Example: the proof projection

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{P(u), \neg P(u) \vee Q(u) \vdash Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} \forall : r
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \hline
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad Q(v) \vdash Q(v) \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \exists : r \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : I \\
 \hline
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)} \forall : I \\
 \hline
 \frac{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} cut
 \end{array}$$

↓

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\vdash P(u), \neg P(u)} \neg : r \\
 \hline
 \frac{\vdash P(u), \neg P(u)}{\neg P(u) \vdash \neg P(u)} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{\neg P(u) \vdash \neg P(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\forall x(\neg P(x) \vee Q(x)) \vdash \neg P(u) \vee Q(u)} \forall : I \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \neg P(u) \vee Q(u)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y), (\neg P(u) \vee Q(u)) \wedge (P(a) \wedge \neg Q(v))} \forall : I
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \quad \frac{Q(v) \vdash Q(v)}{\vdash Q(v), \neg Q(v)} \neg : r \\
 \hline
 \frac{\forall x P(x) \vdash P(a), \vdash Q(v), \neg Q(v)}{\forall x P(x) \vdash Q(v), P(a) \wedge \neg Q(v)} \wedge : r \\
 \hline
 \frac{\forall x P(x) \vdash Q(v), P(a) \wedge \neg Q(v)}{\forall x P(x) \vdash \exists y Q(y), P(a) \wedge \neg Q(v)} \exists : r \\
 \hline
 \frac{\forall x P(x) \vdash \exists y Q(y), P(a) \wedge \neg Q(v)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y), (\neg P(u) \vee Q(u)) \wedge (P(a) \wedge \neg Q(v))} \wedge : r
 \end{array}$$

Example: the proof projection

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{P(u), \neg P(u) \vee Q(u) \vdash Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} \forall : r
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \hline
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad Q(v) \vdash Q(v) \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \exists : r \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : I \\
 \hline
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)} \forall : I \\
 \hline
 \frac{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} cut
 \end{array}$$

$\Downarrow$

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\vdash P(u), \neg P(u)} \neg : r \\
 \hline
 \frac{\vdash P(u), \neg P(u)}{\neg P(u) \vdash \neg P(u)} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{\neg P(u) \vdash \neg P(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\forall x(\neg P(x) \vee Q(x)) \vdash \neg P(u) \vee Q(u)} \forall : I \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \neg P(u) \vee Q(u)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y), (\neg P(u) \vee Q(u)) \wedge (P(a) \wedge \neg Q(v))} \forall : I
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \quad \frac{Q(v) \vdash Q(v)}{\vdash Q(v), \neg Q(v)} \neg : r \\
 \hline
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x) \vdash Q(v), P(a) \wedge \neg Q(v)} \wedge : r \\
 \hline
 \frac{\forall x P(x) \vdash Q(v), P(a) \wedge \neg Q(v)}{\forall x P(x) \vdash \exists y Q(y), P(a) \wedge \neg Q(v)} \exists : r \\
 \hline
 \frac{\forall x P(x) \vdash \exists y Q(y), P(a) \wedge \neg Q(v)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y), (\neg P(u) \vee Q(u)) \wedge (P(a) \wedge \neg Q(v))} \wedge : r
 \end{array}$$

Example: the proof projection

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{P(u), \neg P(u) \vee Q(u) \vdash Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} \forall : r
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \hline
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad Q(v) \vdash Q(v) \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \exists : r \\
 \hline
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : I \\
 \hline
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)} \forall : I \\
 \hline
 \frac{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} cut
 \end{array}$$

⇓

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\vdash P(u), \neg P(u)} \neg : r \\
 \hline
 \frac{\vdash P(u), \neg P(u)}{\neg P(u) \vdash \neg P(u)} \neg : I \quad Q(u) \vdash Q(u) \\
 \hline
 \frac{\neg P(u) \vdash \neg P(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)} \vee : I \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \vee : r \\
 \hline
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\forall x(\neg P(x) \vee Q(x)) \vdash \neg P(u) \vee Q(u)} \forall : I \\
 \hline
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \neg P(u) \vee Q(u)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y), (\neg P(u) \vee Q(u)) \wedge (P(a) \wedge \neg Q(v))} \forall : I
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \quad \frac{Q(v) \vdash Q(v)}{\vdash Q(v), \neg Q(v)} \neg : r \\
 \hline
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x) \vdash Q(v), P(a) \wedge \neg Q(v)} \wedge : r \\
 \hline
 \frac{\forall x P(x) \vdash Q(v), P(a) \wedge \neg Q(v)}{\forall x P(x) \vdash \exists y Q(y), P(a) \wedge \neg Q(v)} \exists : r \\
 \hline
 \frac{\forall x P(x) \vdash \exists y Q(y), P(a) \wedge \neg Q(v)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y), (\neg P(u) \vee Q(u)) \wedge (P(a) \wedge \neg Q(v))} \wedge : r
 \end{array}$$

Example: Herbrand substitutions

$$\Theta_1 = \Theta((\forall x)(\neg P(x) \vee Q(x))) = \{\{x \leftarrow u\}\}.$$

$$\Theta_2 = \Theta((\forall x)P(x)) = \{\{x \leftarrow a\}\}$$

$$\Theta_3 = \Theta((\exists y)Q(y)) = \{\{y \leftarrow v\}\}.$$

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Herbrand substitutions from refutation γ of
 $F = (\neg P(u) \vee Q(u)) \wedge P(a) \wedge \neg Q(v)$:

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$$\Psi(\gamma) = \{u \leftarrow a, v \leftarrow a\}.$$

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Herbrand substitutions from refutation γ of
 $F = (\neg P(u) \vee Q(u)) \wedge P(a) \wedge \neg Q(v)$:

$$\Psi(\gamma) = \{u \leftarrow a, v \leftarrow a\}.$$

Append $\Psi(\gamma)$ to $\Theta_1, \Theta_2, \Theta_3$ and get Herbrand instances.

Example: Herbrand substitutions

$$\Theta_1 = \Theta((\forall x)(\neg P(x) \vee Q(x))) = \{\{x \leftarrow u\}\}.$$

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$$\Psi(\gamma) = \{\{u \leftarrow a, v \leftarrow a\}\}.$$

$$h((\forall x)(\neg P(x) \vee Q(x))) = \Theta_1 \Psi(\gamma) = \{\{x \leftarrow a\}\},$$

$$h((\forall x)P(x)) = \Theta_2 \Psi(\gamma) = \{\{x \leftarrow a\}\},$$

$$h((\exists y)Q(y)) = \Theta_3 \Psi(\gamma) = \{\{y \leftarrow a\}\}.$$

Back to the example proof

φ :

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad \frac{Q(u) \vdash Q(u)}{\neg P(u), \neg P(u) \vee Q(u) \vdash Q(u)} \vee : I \\
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} \forall : r
 \end{array}
 \quad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad \frac{Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \vee : I \\
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : r \\
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash (\exists y) Q(y)} \exists : I \\
 \frac{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash (\exists y) Q(y)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} \forall : I \quad cut
 \end{array}$$

$$h((\forall x)(\neg P(x) \vee Q(x))) = \Theta_1 \Psi(\gamma) = \{\{x \leftarrow a\}\},$$

$$h((\forall x)P(x)) = \Theta_2 \Psi(\gamma) = \{\{x \leftarrow a\}\},$$

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Back to the example proof

φ :

$$\begin{array}{c}
 \frac{P(u) \vdash P(u)}{\neg P(u), P(u) \vdash} \neg : I \quad \frac{Q(u) \vdash Q(u)}{\neg P(u), \neg P(u) \vee Q(u) \vdash Q(u)} \vee : I \\
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u), Q(u)}{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)} \neg : r \\
 \frac{\neg P(u) \vee Q(u) \vdash \neg P(u) \vee Q(u)}{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))} \vee : r \\
 \frac{\neg P(u) \vee Q(u) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))} \exists : r \\
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \exists y(\neg P(u) \vee Q(y))}{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))} \forall : I \\
 \frac{\forall x(\neg P(x) \vee Q(x)) \vdash \forall x \exists y(\neg P(x) \vee Q(y))}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} \forall : r
 \end{array}
 \quad
 \begin{array}{c}
 \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)} \forall : I \\
 \frac{\forall x P(x) \vdash P(a)}{\forall x P(x), \neg P(a) \vdash} \neg : I \quad \frac{Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)} \vee : I \\
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash Q(v)}{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)} \exists : r \\
 \frac{\forall x P(x), \neg P(a) \vee Q(v) \vdash \exists y Q(y)}{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)} \exists : I \\
 \frac{\forall x P(x), \exists y(\neg P(a) \vee Q(y)) \vdash \exists y Q(y)}{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash (\exists y) Q(y)} \forall : I \\
 \frac{\forall x P(x), \forall x \exists y(\neg P(x) \vee Q(y)) \vdash (\exists y) Q(y)}{\forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)} cut
 \end{array}$$

$$h((\forall x)(\neg P(x) \vee Q(x))) = \Theta_1 \Psi(\gamma) = \{\{x \leftarrow a\}\},$$

$$h((\forall x)P(x)) = \Theta_2 \Psi(\gamma) = \{\{x \leftarrow a\}\},$$

$$h((\exists y)Q(y)) = \Theta_3 \Psi(\gamma) = \{\{y \leftarrow a\}\}.$$

$$S: \forall x P(x), \forall x(\neg P(x) \vee Q(x)) \vdash \exists y Q(y)$$

$$H(S): P(a), \neg P(a) \vee Q(a) \vdash Q(a)$$

Soundness

Herbrand sequents can be **extracted from the projections**.
 Substituting them with the substitutions **obtained from the refutation of the characteristic formula** results in the Herbrand sequents obtainable using the original method based on the construction of a CERES normal form.

$$\begin{array}{c}
 \begin{array}{c}
 (\varphi^*\theta_2) \qquad \qquad \qquad (\varphi^*\theta_1) \qquad \qquad \qquad (\gamma) \\
 \Gamma \vdash \Delta, F\theta_2 \qquad \frac{\Gamma \vdash \Delta, F\theta_1 \qquad F\theta_1, \dots, F\theta_n \vdash}{\Gamma, F\theta_2, \dots, F\theta_n \vdash \Delta} cut \\
 \hline
 \Gamma, F\theta_3, \dots, F\theta_n \vdash \Delta \qquad cut + c^*
 \end{array} \\
 \vdots \\
 \frac{(\varphi^*\theta_n) \qquad \Gamma \vdash \Delta, F\theta_n}{\Gamma \vdash \Delta} cut + c^*
 \end{array}$$

Soundness

Theorem

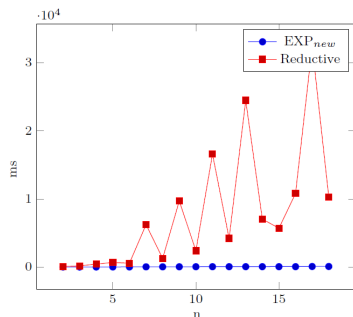
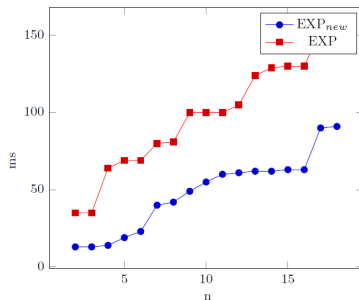
Given a proof with cuts φ , let $H(\varphi^*)$ be the Herbrand sequent extracted from the **projection** φ^* alone. Let Θ be the set of substitutions obtained from a **refutation** of the characteristic formula. Then

$$\times_{\theta \in \Theta} H(\varphi^*)\theta$$

is the Herbrand sequent of the original proof ($\times$ denoting the merge operator on sequents).

Complexity

- ▶ $EXP_{new}(\varphi) \leq EXP(\varphi)$ for all **LK**-proofs φ
- ▶ There exists no constant d s.t. for all **LK**-proofs φ
 $EXP(\varphi) \leq d * EXP_{new}(\varphi)$



Cut-elimination still has a non-elementary computational complexity.

Back to proof theory...

Can a given Herbrand sequent be extracted from a given proof?

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- ▶ traditional cut-elimination is non-confluent

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- ▶ traditional cut-elimination is non-confluent
- ▶ different reduction strategies $\rightarrow$ different cut-free proofs $\rightarrow$ different Herbrand sequents

Back to proof theory...

Can a given Herbrand sequent be extracted from a given proof?

- ▶ traditional cut-elimination is non-confluent
- ▶ different reduction strategies $\rightarrow$ different cut-free proofs $\rightarrow$ different Herbrand sequents
- ▶ CERES simulates Gentzen-style normalization procedures

Herbrand sequents and CERES

$$\begin{array}{c}
 \frac{P\alpha \vdash P\alpha \quad \frac{\frac{Pf\alpha \vdash Pf\alpha \quad Pf^2\alpha \vdash Pf^2\alpha}{Pf\alpha, Pf\alpha \rightarrow Pf^2\alpha \vdash Pf^2\alpha} \rightarrow_I}{P\alpha, P\alpha \rightarrow Pf\alpha, Pf\alpha \rightarrow Pf^2\alpha \vdash Pf^2\alpha} \rightarrow_I}{P\alpha, \forall x(Px \rightarrow Pfx), \forall x(Px \rightarrow Pfx) \vdash Pf^2\alpha} \forall_I, \forall_I \\
 \frac{}{\forall x(Px \rightarrow Pfx), \forall x(Px \rightarrow Pfx) \vdash P\alpha \rightarrow Pf^2\alpha} \rightarrow_r \\
 \frac{}{\forall x(Px \rightarrow Pfx) \vdash P\alpha \rightarrow Pf^2\alpha} c_l \\
 \frac{}{\forall x(Px \rightarrow Pfx) \vdash \forall x(Px \rightarrow Pf^2x)} \forall_r \\
 (\varphi_1)
 \end{array}$$

$$\begin{array}{c}
 \frac{Pf^2a \vdash Pf^2a \quad Pf^4a \vdash Pf^4a}{Pf^2a, Pf^2a \rightarrow Pf^4a \vdash Pf^4a} \rightarrow_I \\
 \frac{Pa \vdash Pa \quad Pf^2a, Pf^2a \rightarrow Pf^4a \vdash Pf^4a}{Pa, Pa \rightarrow Pf^2a, Pf^2a \rightarrow Pf^4a \vdash Pf^4a} \rightarrow_I \\
 \frac{}{Pa, \forall x(Px \rightarrow Pf^2x), \forall x(Px \rightarrow Pf^2x) \vdash Pf^4a} \forall_I, \forall_I \\
 \frac{}{Pa, \forall x(Px \rightarrow Pf^2x) \vdash Pf^4a} c_l \\
 \frac{}{Pa, \forall x(Px \rightarrow Pf^2x) \vdash Pf^4a \vee Pg^4a} \vee_r + c_r \\
 \frac{}{Pa, \forall x(Px \rightarrow Pf^2x) \vdash \exists z(Pf^4z \vee Pg^4z)} \exists_r \\
 \frac{}{Pa, \forall x(Px \rightarrow Pf^2x), \forall x(Px \rightarrow Pgx) \vdash \exists z(Pf^4z \vee Pg^4z)} w_l \\
 \frac{}{Pa, \forall x(Px \rightarrow Pfx), \forall x(Px \rightarrow Pgx) \vdash \exists z(Pf^4z \vee Pg^4z)} cut
 \end{array}$$

What is the Herbrand sequent of φ ?

φ is an **LK**-proof of

$$S: Pa, \forall x(Px \rightarrow Pfx), \forall x(Px \rightarrow Pgx) \vdash \exists z(Pf^4z \vee Pg^4z).$$

What is the Herbrand sequent of φ ?

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$$S: Pa, \forall x(Px \rightarrow Pfx), \forall x(Px \rightarrow Pgx) \vdash \exists z(Pf^4z \vee Pg^4z).$$

A Herbrand sequent of S is $H(S)$:

$$Pa,$$

$$Pa \rightarrow Pga,$$

$$Pga \rightarrow Pg^2a,$$

$$Pg^2a \rightarrow Pg^3a,$$

$$Pg^3a \rightarrow Pg^4a \vdash Pf^4a \vee Pg^4a$$

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Can we obtain $H(S)$ from φ ?

What is the Herbrand sequent of φ ?

φ is an **LK**-proof of

$$S: Pa, \forall x(Px \rightarrow Pfx), \forall x(Px \rightarrow Pgx) \vdash \exists z(Pf^4z \vee Pg^4z).$$

A Herbrand sequent of S is $H(S)$:

$$Pa,$$

$$Pa \rightarrow Pga,$$

$$Pga \rightarrow Pg^2a,$$

$$Pg^2a \rightarrow Pg^3a,$$

$$Pg^3a \rightarrow Pg^4a \vdash Pf^4a \vee Pg^4a$$

Can we obtain $H(S)$ from φ ? No.

Running CERES on this example yields...

characteristic formula: $Pa \wedge (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a \wedge (\neg P\alpha \vee Pf^2\alpha)$

refutation with unifiers $\{\alpha \leftarrow a\}, \{\alpha \leftarrow f^2a\}$

projection φ^* :

$$\begin{array}{c}
 \frac{P\alpha \vdash P\alpha}{\vdash P\alpha, \neg P\alpha} \neg : r \quad \frac{Pf\alpha \vdash Pf\alpha \quad Pf^2\alpha \vdash Pf^2\alpha}{Pf\alpha, Pf\alpha \rightarrow Pf^2\alpha \vdash Pf^2\alpha} \rightarrow : l \\
 \hline
 \frac{}{P\alpha \rightarrow Pf\alpha, Pf\alpha \rightarrow Pf^2\alpha \vdash \neg P\alpha, Pf^2\alpha} \rightarrow : l \\
 \hline
 \frac{}{P\alpha \rightarrow Pf\alpha, Pf\alpha \rightarrow Pf^2\alpha \vdash \neg P\alpha \vee Pf^2\alpha} \vee : r \\
 \hline
 \frac{\forall x(Px \rightarrow Pfx), \forall x(Px \rightarrow Pfx) \vdash \neg P\alpha \vee Pf^2\alpha}{\forall x(Px \rightarrow Pfx) \vdash \neg P\alpha \vee Pf^2\alpha} \forall I, \forall I \\
 \hline
 \frac{}{\forall x(Px \rightarrow Pfx) \vdash \neg P\alpha \vee Pf^2\alpha} cI \\
 \hline
 (\varphi_1^*) \\
 \frac{Pf^2a \vdash Pf^2a}{\vdash Pf^2a, \neg Pf^2a} \neg : r \quad \frac{Pf^4a \vdash Pf^4a}{\vdash Pf^4a, \neg Pf^4a} \neg : r \\
 \hline
 \frac{\vdash Pf^2a \vee \neg Pf^2a}{\vdash Pf^2a \vee \neg Pf^2a} \vee : r \quad \frac{\vdash Pf^4a, \neg Pf^4a}{\vdash Pf^4a, \neg Pf^4a} \neg : r \\
 \hline
 \frac{Pa \vdash Pa \quad \vdash Pf^4a, (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a}{\vdash Pf^4a, (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a} \wedge : r \\
 \hline
 \frac{Pa \vdash Pf^4a, Pa \wedge (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a}{Pa \vdash Pf^4a \vee Pg^4a, Pa \wedge (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a} \wedge : r \\
 \hline
 \frac{Pa \vdash Pf^4a \vee Pg^4a, Pa \wedge (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a}{Pa \vdash \exists z(Pf^4z \vee Pg^4z), Pa \wedge (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a} \exists r \\
 \hline
 \frac{Pa \vdash \exists z(Pf^4z \vee Pg^4z), Pa \wedge (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a}{Pa, \forall x(Px \rightarrow Pgx) \vdash \exists z(Pf^4z \vee Pg^4z), Pa \wedge (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a} wI \\
 \hline
 \frac{Pa, \forall x(Px \rightarrow Pgx) \vdash \exists z(Pf^4z \vee Pg^4z), Pa \wedge (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a}{Pa, \forall x(Px \rightarrow Pfx), \forall x(Px \rightarrow Pgx) \vdash \exists z(Pf^4z \vee Pg^4z), F} \wedge : r
 \end{array}$$

Running CERES on this example yields...

characteristic formula: $Pa \wedge (Pf^2a \vee \neg Pf^2a) \wedge \neg Pf^4a \wedge (\neg P\alpha \vee Pf^2\alpha)$

refutation with unifiers $\{\alpha \leftarrow a\}, \{\alpha \leftarrow f^2a\}$

projection φ^* :

The Herbrand sequent from the projection is:

$$H(\varphi^*): Pa, P\alpha \rightarrow Pf\alpha, Pf\alpha \rightarrow Pf^2\alpha \vdash Pf^4a \vee Pg^4a.$$

Herbrand sequent of φ is $H(\varphi^*)\{\alpha \leftarrow a\} \circ H(\varphi^*)\{\alpha \leftarrow f^2a\}$:

$$Pa, Pa \rightarrow Pfa, Pf^2a \rightarrow Pf^3a, Pfa \rightarrow Pf^2a, Pf^3a \rightarrow Pf^4a \vdash Pf^4a \vee Pg^4a.$$

There is no substitution on α that results in

$$Pa, Pa \rightarrow Pga, Pga \rightarrow Pg^2a, Pg^2a \rightarrow Pg^3a, Pg^3 \rightarrow Pg^4a \vdash Pf^4a \vee Pg^4a.$$

Extensions

- ▶ CERES and interpolation,

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Theoretical Computer Science, Volume 1085, 2026, 116212, ISSN
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- ▶ CERES and macros.

$$\left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v)$$

CERES and ...?



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